Aula 8 — Funções Compostas

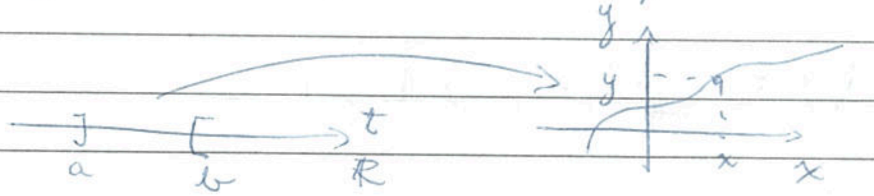
Seja $z = f(x, y)$ ~~com x e y variáveis independentes~~ uma função diferenciável em uma região R

$$\cancel{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$

do plano. Se x e y são funções da mesma variável independente t,

$$x = \varphi(t) \quad e \quad y = \psi(t)$$

z é uma função composta de t



$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

eq. paramétricas

$$z = f(x, y) = f(\varphi(t), \psi(t)) = F(t)$$

Derivação

Teorema: Regra da cadeia

$z = f(x, y)$ é diferenciável em $(x_0, y_0) \in D(f)$ e $x = \varphi(t)$ e $y = \psi(t)$ são funções ~~deriváveis de~~ dif em t_0, de modo que $x(t_0) = x_0$ e $y(t_0) = y_0$. Então

$$\frac{dF}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0)$$

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Pela definição,

$$\frac{dF}{dt}(t_0) = \lim_{\Delta t \to 0} \frac{F(t_0+\Delta t) - F(t_0)}{\Delta t}$$

Mas $F(t_0+\Delta t) - F(t_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

onde $\begin{cases} \Delta x = x(t_0+\Delta t) - x(t_0) \\ \Delta y = y(t_0+\Delta t) - y(t_0) \end{cases}$

Como f é diferenciável em (x_0, y_0), temos

$f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) =$

$$= \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y +$$

$$+ \Delta x\, h_1(\Delta x, \Delta y) + \Delta y\, h_2(\Delta x, \Delta y)$$

onde $\lim\limits_{\Delta x, \Delta y \to (0,0)} h_1 = \lim h_2 = 0$

Portanto,

$$\frac{dF}{dt}(t_0) = \lim_{\Delta t \to 0} \left[\frac{\partial f}{\partial x}(x_0, y_0) \frac{\Delta x}{\Delta t} + \right.$$

$$\left. + \frac{\partial f}{\partial y}(x_0, y_0)\frac{\Delta y}{\Delta t} + \frac{\Delta x}{\Delta t} h_1 + \frac{\Delta y}{\Delta t} h_2 \right]$$

~~Como~~ Lembrando que

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}(t_0)$$

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}(t_0)$$

e como x e y são diferenciáveis, também são contínuas:

$$\lim_{\Delta t \to 0} \Delta x = \lim_{\Delta t \to 0} [x(t_0 + \Delta t) - x(t_0)] = 0$$

$$\lim_{\Delta t \to 0} \Delta y = \lim_{\Delta t \to 0} [y(t_0 + \Delta t - y(t_0)] = 0$$

então, $\lim_{\Delta t \to 0} h_1 = \lim_{\Delta t \to 0} h_2 = 0$

e obtemos

$$\boxed{\frac{dF}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0)}$$

Exemplo: Sejam $f(x, y) = 2x + 5y - 3$, $x(t) = 2t$ e $y(t) = 3t - 1$.

$F(t) = 2(2t) + 5(3t - 1) - 3$

$F(t) = 19t - 8$

a) Cálculo de $\frac{dF}{dt}$ diretamente: $\frac{dF}{dt} = 19$

TAMOIO

b) Cálculo de $\frac{dF}{dt}$ pela regra da cadeia:

$\frac{\partial f}{\partial x} = 2$, $\frac{\partial f}{\partial y} = \cancel{3}5$

$\frac{dx}{dt} = 2$, $\frac{dy}{dt} = 3$

$$\frac{dF}{dt} = 2.2 + 5.3 = 19$$

Exemplo: Sejam $f(x,y) = x^2 + 3y - 5$, $x(t) = e^t$ e $y(t) = t^3$

$$F(t) = (e^t)^2 + 3t^3 - 5 = e^{2t} + 3t^3 - 5$$

a) Cálculo de $\frac{dF}{dt}$ diretamente: $\frac{dF}{dt} = 2e^{2t} + 9t^2$

b) Cálculo de $\frac{dF}{dt}$ pela regra da cadeia:

$\frac{\partial f}{\partial x} = 2x = 2e^t$, $\frac{\partial f}{\partial y} = 3$

$\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = 3t^2$

$$\frac{dF}{dt} = 2\cancel{x}e^t e^t + 3.3t^2 = 2e^{2t} + 9t^2$$

Exercício: Seja $z = 3x^2 + 5y^2 - 4xy$ e $x = tg\,2t$ e $y = sen^2 t$. Encontrar $\frac{dz}{dt}$ pela regra da cadeia.

$$\frac{\partial z}{\partial x} = 6x - 4y, \quad \frac{\partial z}{\partial y} = 10y - 4x$$

$$\frac{dx}{dt} = 2\sec^2 2t, \quad \frac{dy}{dt} = 2\,sen\,t \cos t = sen\,2t$$

$$\frac{dz}{dt} = (6x - 4y)\,2\sec^2 2t + (10y - 4x)\,sen\,2t$$

Caso de duas variáveis independentes

$$x = \varphi(t,s) \quad e \quad y = \psi(t,s)$$

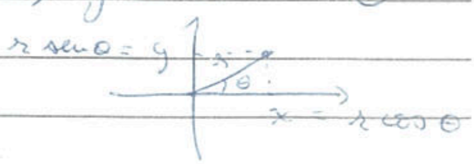
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$ derivadas parciais!

e $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Exemplos: Coordenadas polares

Seja $z = f(x,y)$ e $x = r\cos\theta$, $y = r\,sen\,\theta$

Calcular $\frac{\partial z}{\partial r}$ e $\frac{\partial z}{\partial \theta}$ e



verificar a relação:

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

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$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\operatorname{sen}\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x} r\operatorname{sen}\theta + \frac{\partial z}{\partial y} r\cos\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\operatorname{sen}\theta\cos\theta +$$

$$+\left(\frac{\partial z}{\partial y}\right)^2\operatorname{sen}^2\theta$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2\operatorname{sen}^2\theta \cancel{+} - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} r^2\operatorname{sen}\theta\cos\theta +$$

$$+\left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \;/\!/$$

Generalização

$y = f(x_1, x_2, \dots, x_n)$

$x_1 = \varphi_1(t_1, t_2, \dots, t_m)$
$x_2 = \varphi_2(t_1, t_2, \dots, t_m)$
$\vdots$
$x_n = \varphi_n(t_1, t_2, \dots, t_m)$

$$\frac{\partial y}{\partial t_1} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \dots + \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial y}{\partial t_m} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \dots + \frac{\partial y}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Diferenciais de funções compostas

Se $z = f(x, y)$ com x e y variáveis independentes

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Se $x = \varphi(t, s)$ e $y = \psi(t, s)$.

$$z = f(x, y) = f(\varphi(t, s), \psi(t, s)) = F(t, s)$$

e $dz = \frac{\partial z}{\partial t}dt + \frac{\partial z}{\partial s}ds$

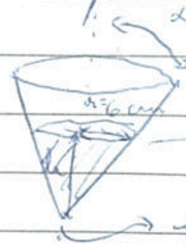
mas $dx = \frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial s}ds$ e $dy = \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial s}ds$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)dt +$$

$$+\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)ds = \frac{\partial z}{\partial t}dt + \frac{\partial z}{\partial s}ds = dz$$

ou $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

Exemplo: A água escoa à razão de $18\pi\ cm^3/s$



→ funil cônico

→ água

→ furo

qual a velocidade com que baixa a água?

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial t} = \frac{2\pi r h}{3}\frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}$$

$$= \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = 18\pi\ cm^3/s$$

$$tg\,\alpha = \frac{r}{h} \quad \text{ou} \quad r = h\,tg\,\alpha \Rightarrow \frac{dr}{dt} = tg\,\alpha\frac{dh}{dt}$$

$$\alpha = 30° \Rightarrow r = \frac{\sqrt{3}}{3}h \ \text{e}\ \frac{dr}{dt} = \frac{\sqrt{3}}{3}\frac{dh}{dt}$$

$$\Rightarrow h = \sqrt{3}\,r$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(\frac{2\cancel{\sqrt{3}}r^2\cancel{\sqrt{3}}}{\cancel{3}} + r^2\right)\frac{dh}{dt} = \frac{\pi}{\cancel{3}}\cancel{3}r^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt} = \frac{1}{\cancel{\pi}r^2}18\cancel{\pi} = \frac{18}{6^2}$$

$$= \frac{18}{36} = \frac{1}{2}\ cm/s \quad (r = 6\,cm)$$