On random walks with dependence of its past
ERMAC 2024–SBMAC–Regional 8

Cristian Coletti - UFABC - cristian.coletti@ufabc.edu.br

Joint work with Renato Gava (UFSCAR), Gunter Schütz (ICS), Ioannis Papageorgiou (UFABC), Denis Araujo Luiz (UFABC) and Lucas de Lima (UFABC - USP)

28 de agosto de 2024

Outline

- Introduction
- The ERW model
- Main results
- Two repelling Random Walks
- Idea of the proof

Introduction

- In the first part of the talk, we consider the so called elephant random walk introduced by Schutz and Trimper and a related model, the D.E.R.W.

Introduction

- In the first part of the talk, we consider the so called elephant random walk introduced by Schutz and Trimper and a related model, the D.E.R.W.

- In the ERW the walker remember the whole past. Thus, the next step always depends on the whole past.

Introduction

- In the first part of the talk, we consider the so called elephant random walk introduced by Schutz and Trimper and a related model, the D.E.R.W.
- In the ERW the walker remember the whole past. Thus, the next step always depends on the whole past.
- Martingale theory allows to prove many limit theorem for this model and its generalization.

Recent Works

- Baur, E., Bertoin, J. Elephant random walks and their connection to Polya-type urns. Phys. Rev. E 49 052134 (2016).
- Bercu, B. A martingale approach for the elephant random walk. J. Phys. A: Math. Theor. 51 015201 (2017).
- Bercu, B., Laulin, L. On the Multi-dimensional Elephant Random Walk. J. Stat. Phys. 175(6) (2019), 1146?1163.
- C., C. F., Gava, R., and Schutz, G. M. Central Limit Theorem for the Elephant Random Walk. J. Math. Phys. 58(5) (2017).
- C., C. F., Gava, R., and Schutz, G. M. A strong invariance principle for the elephant random walk. J. Stat. Mech. Theory Exp. 12 (2017), 123207.

Recent Works

- Vazquez, V. On the almost sure central limit theorem for the elephant random walk. Journal of Physics A: Mathematical and Theoretical. 52(47) (2019).
- Bercu, B. and Vazquez, V. New insights on the minimal random walk. Preprint (2021).
- Bercu, B. and Laulin, L. How to estimate the memory of the Elephant random walk. Preprint.
- Gut, A. and Stadtmuller, U. The elephant random walk with gradually increasing memory. Preprint.

The Elephant Random Walk

- It was introduced by Schütz and Trimper in 2004.

The Elephant Random Walk

- It was introduced by Schütz and Trimper in 2004.
- The elephant departs from the origin. Denoting by X_n its position by time n we have $X_0 = 0$. The first step is given as follows: $\mathbb{P}(X_1 = 1) = q = 1 - \mathbb{P}(X_1 = -1)$.

The Elephant Random Walk

- It was introduced by Schütz and Trimper in 2004.
- The elephant departs from the origin. Denoting by X_n its position by time n we have $X_0 = 0$. The first step is given as follows: $\mathbb{P}(X_1 = 1) = q = 1 - \mathbb{P}(X_1 = -1)$.
- For $n \geq 1$ let

$$X_{n+1} = X_n + \eta_{n+1}$$

where $\eta_{n+1} = \pm 1$ is a r.v. defined as follows.

The Elephant Random Walk

- It was introduced by Schütz and Trimper in 2004.
- The elephant departs from the origin. Denoting by X_n its position by time n we have $X_0 = 0$. The first step is given as follows: $\mathbb{P}(X_1 = 1) = q = 1 - \mathbb{P}(X_1 = -1)$.
- For $n \geq 1$ let
$$X_{n+1} = X_n + \eta_{n+1}$$
where $\eta_{n+1} = \pm 1$ is a r.v. defined as follows.
- At time $n + 1$ a number $n' \in \{1, 2, \ldots, n\}$ is chosen at random with probability $1/n$.

The Elephant Random Walk

- It was introduced by Schütz and Trimper in 2004.
- The elephant departs from the origin. Denoting by X_n its position by time n we have $X_0 = 0$. The first step is given as follows: $\mathbb{P}(X_1 = 1) = q = 1 - \mathbb{P}(X_1 = -1)$.
- For $n \geq 1$ let
$$X_{n+1} = X_n + \eta_{n+1}$$
where $\eta_{n+1} = \pm 1$ is a r.v. defined as follows.
- At time $n+1$ a number $n' \in \{1, 2, \ldots, n\}$ is chosen at random with probability $1/n$.
- $$\eta_{n+1} = \eta_{n'} \text{ w.p. } p \text{ and } \eta_{n+1} = -\eta_{n'} \text{ w.p. } 1-p.$$

The Elephant Random Walk

- It was introduced by Schütz and Trimper in 2004.
- The elephant departs from the origin. Denoting by X_n its position by time n we have $X_0 = 0$. The first step is given as follows: $\mathbb{P}(X_1 = 1) = q = 1 - \mathbb{P}(X_1 = -1)$.
- For $n \geq 1$ let
$$X_{n+1} = X_n + \eta_{n+1}$$
where $\eta_{n+1} = \pm 1$ is a r.v. defined as follows.
- At time $n + 1$ a number $n' \in \{1, 2, \ldots, n\}$ is chosen at random with probability $1/n$.
-

$$\eta_{n+1} = \eta_{n'} \text{ w.p. } p \text{ and } \eta_{n+1} = -\eta_{n'} \text{ w.p. } 1 - p.$$

The Elephant Random Walk

- It was introduced by Schütz and Trimper in 2004.
- The elephant departs from the origin. Denoting by X_n its position by time n we have $X_0 = 0$. The first step is given as follows: $\mathbb{P}(X_1 = 1) = q = 1 - \mathbb{P}(X_1 = -1)$.
- For $n \geq 1$ let
$$X_{n+1} = X_n + \eta_{n+1}$$
where $\eta_{n+1} = \pm 1$ is a r.v. defined as follows.
- At time $n+1$ a number $n' \in \{1, 2, \ldots, n\}$ is chosen at random with probability $1/n$.
- $$\eta_{n+1} = \eta_{n'} \text{ w.p. } p \text{ and } \eta_{n+1} = -\eta_{n'} \text{ w.p. } 1-p.$$

First and second moments

- Schütz and Trimper (2004) showed that

$$\mathbb{E}[X_n] = (2q-1)\frac{\Gamma(n+(2p-1))}{\Gamma(2p)\Gamma(n)} \sim \frac{2q-1}{\Gamma(2p)} n^{2p-1}.$$

First and second moments

- Schütz and Trimper (2004) showed that

$$\mathbb{E}[X_n] = (2q-1)\frac{\Gamma(n+(2p-1))}{\Gamma(2p)\Gamma(n)} \sim \frac{2q-1}{\Gamma(2p)} n^{2p-1}.$$

-

$$\mathbb{E}[X_n^2] \sim \begin{cases} \frac{n}{3-4p}, & \text{if} \quad p < 3/4 \\ n \log n, & \text{if} \quad p = 3/4 \\ \frac{n^{4p-2}}{(4p-3)(\Gamma(4p-2))} & \text{if} \quad p > 3/4 \end{cases}$$

First and second moments

- Schütz and Trimper (2004) showed that

$$\mathbb{E}[X_n] = (2q-1)\frac{\Gamma(n+(2p-1))}{\Gamma(2p)\Gamma(n)} \sim \frac{2q-1}{\Gamma(2p)} n^{2p-1}.$$

-

$$\mathbb{E}[X_n^2] \sim \begin{cases} \frac{n}{3-4p}, & \text{if} \quad p < 3/4 \\ n \log n, & \text{if} \quad p = 3/4 \\ \frac{n^{4p-2}}{(4p-3)(\Gamma(4p-2))} & \text{if} \quad p > 3/4 \end{cases}$$

Law of large numbers and central limit theorem (C, Gava and Schutz)

- Thm: Let $(X_n)_{n\geq 1}$ be the ERW. Then for any value of q and $p \in [0,1)$

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \text{ a.s.}$$

Law of large numbers and central limit theorem (C, Gava and Schutz)

- Thm: Let $(X_n)_{n\geq 1}$ be the ERW. Then for any value of q and $p \in [0, 1)$

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \text{ a.s.}$$

- Thm: Let $(X_n)_{n\geq 1}$ be the ERW.
(a) If $p < 3/4$, then

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} N(0, \frac{1}{3-4p}).$$

(b) If $p = 3/4$, then

$$\frac{X_n}{\sqrt{n \log n}} \xrightarrow{d} N(0, 1).$$

Almost sure convergence (C.G.S.)

- Thm: Let $(X_n)_{n\geq 1}$ be the ERW. If $3/4 < p \leq 1$, then

$$\frac{X_n}{n^{2p-1}} \to M \text{ a.s. },$$

where M is a non-degenerate mean zero random variable, but not a normal r.v..

Recurrence – Transience for the ERW

Theorem (I. Papageorgiou, C)

Let $(X_n)_{n\geq 0}$ be the ERW with full memory. Then, if $p \leq 3/4$ the ERW is recurrent.

Recurrence – Transience for the ERW

Theorem (I. Papageorgiou, C)

Let $(X_n)_{n \geq 0}$ *be the ERW with full memory. Then, if* $p \leq 3/4$ *the ERW is recurrent.*

Remark Indeed, if $p < 1/6$ the ERW is positive recurrent.

Recurrence – Transience for the ERW

Theorem (I. Papageorgiou, C)

Let $(X_n)_{n \geq 0}$ be the ERW with full memory. Then, if $p > 3/4$ the ERW is transient.

Strong approximations: Motivation

- The CLT says that if $X_1, \dots, X_n, \dots$ are i.i.d. rv's with mean μ and variance σ^2 then

$$\lim_{n \to +\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right) = \Phi(x), \tag{1}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$.

Strong approximations: Motivation

- The CLT says that if $X_1, \ldots, X_n, \ldots$ are i.i.d. rv's with mean μ and variance σ^2 then

$$\lim_{n \to +\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right) = \Phi(x), \tag{1}$$

  where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$.
- How good is this approximation?

Strong approximations: Motivation

- The CLT says that if $X_1, \ldots, X_n, \ldots$ are i.i.d. rv's with mean μ and variance σ^2 then

$$\lim_{n \to +\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \le x\right) = \Phi(x), \tag{1}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$.

- How good is this approximation? Under the same hypothesis we have

$$\lim_{n \to +\infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \le x\right) - \Phi(x) \right| = 0. \tag{2}$$

Strong approximations: Motivation

Strong approximations: Motivation

- Then we may approximate $\mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x_n\right)$ by $\Phi(x_n)$.

Strong approximations: Motivation

- Then we may approximate $\mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x_n\right)$ by $\Phi(x_n)$. If $\rho := \mathbb{E}\left(|X_1 - \mu|\right) < +\infty$, the error is bounded by

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right) - \Phi(x) \right| \leq \frac{\rho}{2\sigma^3\sqrt{n}} \qquad (3)$$

Strong approximations: Motivation

- Then we may approximate $\mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x_n\right)$ by $\Phi(x_n)$. If $\rho := \mathbb{E}\left(|X_1 - \mu|\right) < +\infty$, the error is bounded by

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right) - \Phi(x) \right| \leq \frac{\rho}{2\sigma^3\sqrt{n}} \qquad (3)$$

- Is it possible to improve this?

Strong approximations: Motivation

- Then we may approximate $\mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x_n\right)$ by $\Phi(x_n)$. If $\rho := \mathbb{E}\left(|X_1 - \mu|\right) < +\infty$, the error is bounded by

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right) - \Phi(x) \right| \leq \frac{\rho}{2\sigma^3\sqrt{n}} \tag{3}$$

- Is it possible to improve this?
- Yes, Strong Invariance Principles!!

What is a strong invariance principle?

- It's a limit theorem concerning strong approximation for partial sums process of some random sequence or field by a (multiparameter) Wiener process.

What is a strong invariance principle?

- It's a limit theorem concerning strong approximation for partial sums process of some random sequence or field by a (multiparameter) Wiener process.
- In 1964, Strassen proved that if $(X_j)_{j\geq 1}$ are i.i.d. r.v.'s with zero mean and variance σ^2 then it is possible to construct a sequence $(Z_j)_{j\geq 1}$ of centred Gaussian r.v.'s with variance σ^2 such that

What is a strong invariance principle?

- It's a limit theorem concerning strong approximation for partial sums process of some random sequence or field by a (multiparameter) Wiener process.
- In 1964, Strassen proved that if $(X_j)_{j\geq 1}$ are i.i.d. r.v.'s with zero mean and variance σ^2 then it is possible to construct a sequence $(Z_j)_{j\geq 1}$ of centred Gaussian r.v.'s with variance σ^2 such that

$$\sup_{1\leq k\leq n}\left|\sum_{k=1}^{n}(Z_j - X_j)\right| = o\left(b_n\right) \text{ a.s., when } n \to +\infty. \qquad (4)$$

where $b_n = \sqrt{n\log\left(\log\left(n\right)\right)}$.

What is a strong invariance principle?

- It's a limit theorem concerning strong approximation for partial sums process of some random sequence or field by a (multiparameter) Wiener process.
- In 1964, Strassen proved that if $(X_j)_{j\geq 1}$ are i.i.d. r.v.'s with zero mean and variance σ^2 then it is possible to construct a sequence $(Z_j)_{j\geq 1}$ of centred Gaussian r.v.'s with variance σ^2 such that

$$\sup_{1\leq k\leq n}\left|\sum_{k=1}^{n}(Z_j - X_j)\right| = o\left(b_n\right) \text{ a.s., when } n\to+\infty. \qquad (4)$$

  where $b_n = \sqrt{n\log\left(\log\left(n\right)\right)}$.
- What can be said in the case of correlated r.v.'s?

Invariance principle. C, Gava and Schütz

- Thm: Let $(X_n)_{n\geq 1}$ be the ERW with $p \leq 3/4$ and let $\{W_t\}_{t\geq 0}$ be B.M. Then, there exists a common probability space to X_n and W_t s.t.
  a) If $p < 3/4$, then

$$\left|\sqrt{3-4p}\frac{X_n}{n^{2p-1}} - W\left(n^{3-4p}\right)\right| = o(\sqrt{n^{3-4p}\log\log n}) \quad \text{a.s.}$$

  b) If $p = 3/4$, then

$$\left|\frac{X_n}{\sqrt{n}} - W(\log n)\right| = o(\sqrt{\log n \times \log\log\log n}) \quad \text{a.s.}$$

Law of iterated logarithm (C, Gava and Schütz)

- Corollary: Let $(X_n)_{n\geq 1}$ be the ERW and let $p \leq 3/4$.

  a) If $p < 3/4$, then

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{n \log \log n}} = \sqrt{\frac{2}{3-4p}} \text{ a.s.}$$

  b) If $p = 3/4$, then

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{n \log n} \times \sqrt{\log \log \log n}} = \sqrt{2} \text{ a.s..}$$

Martingales

- A filtered space is a probability space together with a filtration $\{\mathcal{F}_n\}$ i.e. an increasing family of sub-σ-algebra of $\mathcal{F}$.

Martingales

- A filtered space is a probability space together with a filtration $\{\mathcal{F}_n\}$ i.e. an increasing family of sub-σ-algebra of $\mathcal{F}$.
- An stochastic process (X_n) is a martingale relative to $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ if

Martingales

- A filtered space is a probability space together with a filtration $\{\mathcal{F}_n\}$ i.e. an increasing family of sub-σ-algebra of $\mathcal{F}$.
- An stochastic process (X_n) is a martingale relative to $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ if

Martingales

- A filtered space is a probability space together with a filtration $\{\mathcal{F}_n\}$ i.e. an increasing family of sub-σ-algebra of $\mathcal{F}$.
- An stochastic process (X_n) is a martingale relative to $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ if
- (X_n) is adapted to $\{\mathcal{F}_n\}$, i.e. for each n, X_n is $\mathcal{F}_n$- measurable.

Martingales

- A filtered space is a probability space together with a filtration $\{\mathcal{F}_n\}$ i.e. an increasing family of sub-σ-algebra of $\mathcal{F}$.
- An stochastic process (X_n) is a martingale relative to $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ if
- (X_n) is adapted to $\{\mathcal{F}_n\}$, i.e. for each n, X_n is $\mathcal{F}_n$- measurable.
- $\mathbb{E}\left(|X_n|\right) < +\infty$ for each n.

Martingales

- A filtered space is a probability space together with a filtration $\{\mathcal{F}_n\}$ i.e. an increasing family of sub-σ-algebra of $\mathcal{F}$.
- An stochastic process (X_n) is a martingale relative to $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ if
- (X_n) is adapted to $\{\mathcal{F}_n\}$, i.e. for each n, X_n is $\mathcal{F}_n$- measurable.
- $\mathbb{E}(|X_n|) < +\infty$ for each n.
- $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ a.s. for each n.

Martingale: Intuition through gambling

- Conditional expectation encodes the intuitive idea of taking the expected value of X given the information we have at our disposal i.e. the σ–algebra $\mathcal{F}_n$.

Martingale: Intuition through gambling

- Conditional expectation encodes the intuitive idea of taking the expected value of X given the information we have at our disposal i.e. the σ–algebra $\mathcal{F}_n$.
- A σ-algebra is a mathematical object which encode the notion of information.

Martingale: Intuition through gambling

- Conditional expectation encodes the intuitive idea of taking the expected value of X given the information we have at our disposal i.e. the σ–algebra $\mathcal{F}_n$.
- A σ-algebra is a mathematical object which encode the notion of information.
- The increasing sub-σ-algebra $\{\mathcal{F}_n\}$ describe the play up to the n–th trial.

Martingale: Intuition through gambling

- Conditional expectation encodes the intuitive idea of taking the expected value of X given the information we have at our disposal i.e. the σ–algebra $\mathcal{F}_n$.
- A σ-algebra is a mathematical object which encode the notion of information.
- The increasing sub-σ-algebra $\{\mathcal{F}_n\}$ describe the play up to the n–th trial.
- The variables $X_0, X_1, \ldots, X_n, \ldots$ record your capital which are summable and successively measurable over the $\mathcal{F}'$s.

Martingale: Intuition through gambling

- Conditional expectation encodes the intuitive idea of taking the expected value of X given the information we have at our disposal i.e. the σ–algebra $\mathcal{F}_n$.
- A σ-algebra is a mathematical object which encode the notion of information.
- The increasing sub-σ-algebra $\{\mathcal{F}_n\}$ describe the play up to the n–th trial.
- The variables $X_0, X_1, \ldots, X_n, \ldots$ record your capital which are summable and successively measurable over the $\mathcal{F}'$s.
- The game is fair if

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n.$$

A Martingale

- Put

$$a_1 = 1 \text{ and } a_n = \prod_{j=1}^{n-1} \left(1 + \frac{(2p-1)}{j}\right) \text{ for } n \geq 2$$

A Martingale

- Put

$$a_1 = 1 \text{ and } a_n = \prod_{j=1}^{n-1} \left(1 + \frac{(2p-1)}{j}\right) \text{ for } n \geq 2$$

- Define the filtration $\mathcal{F}_n = \sigma(\eta_1, \ldots, \eta_n)$ and $M_n = \frac{X_n - \mathbb{E}[X_n]}{a_n}$ for $n \geq 1$. We claim that $\{M_n\}_{n \geq 1}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n \geq 1}$.

A Martingale

$$\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] = \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\mathbb{E}\left[\eta_{n+1} \mid \mathcal{F}_n\right] - \mathbb{E}\left[\eta_{n+1}\right]}{a_{n+1}}$$

A Martingale

$$\begin{aligned}\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] &= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\mathbb{E}\left[\eta_{n+1} \mid \mathcal{F}_n\right] - \mathbb{E}\left[\eta_{n+1}\right]}{a_{n+1}} \\ &= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{(2p-1)\frac{X_n}{n} - (2p-1)\frac{\mathbb{E}[X_n]}{n}}{a_{n+1}}\end{aligned}$$

A Martingale

$$\begin{aligned}
\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] &= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\mathbb{E}\left[\eta_{n+1} \mid \mathcal{F}_n\right] - \mathbb{E}\left[\eta_{n+1}\right]}{a_{n+1}} \\
&= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{(2p-1)\frac{X_n}{n} - (2p-1)\frac{\mathbb{E}[X_n]}{n}}{a_{n+1}} \\
&= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\frac{(2p-1)}{n}(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}}
\end{aligned}$$

A Martingale

$$
\begin{aligned}
\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] &= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\mathbb{E}\left[\eta_{n+1} \mid \mathcal{F}_n\right] - \mathbb{E}\left[\eta_{n+1}\right]}{a_{n+1}} \\
&= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{(2p-1)\frac{X_n}{n} - (2p-1)\frac{\mathbb{E}[X_n]}{n}}{a_{n+1}} \\
&= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\frac{(2p-1)}{n}\left(X_n - \mathbb{E}\left[X_n\right]\right)}{a_{n+1}} \\
&= \left(X_n - \mathbb{E}\left[X_n\right]\right)\frac{\left(1 + \frac{(2p-1)}{n}\right)}{a_{n+1}}
\end{aligned}
$$

A Martingale

$$
\begin{aligned}
\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] &= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\mathbb{E}\left[\eta_{n+1} \mid \mathcal{F}_n\right] - \mathbb{E}\left[\eta_{n+1}\right]}{a_{n+1}} \\
&= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{(2p-1)\frac{X_n}{n} - (2p-1)\frac{\mathbb{E}[X_n]}{n}}{a_{n+1}} \\
&= \frac{(X_n - \mathbb{E}\left[X_n\right])}{a_{n+1}} + \frac{\frac{(2p-1)}{n}\left(X_n - \mathbb{E}\left[X_n\right]\right)}{a_{n+1}} \\
&= (X_n - \mathbb{E}\left[X_n\right]) \frac{\left(1 + \frac{(2p-1)}{n}\right)}{a_{n+1}} \\
&= M_n.
\end{aligned}
$$

Dynamic elephant random walk

- (Pure) Dynamic random walk.
  - Let (E, Σ, μ) be a complete probability space.

Dynamic elephant random walk

- (Pure) Dynamic random walk.
  - Let (E, Σ, μ) be a complete probability space.
  - Let $T : E \to E$ be a one-to-one onto map such that T and T^{-1} are both measurable. Assume also that T is measure preserving, i.e.,

Dynamic elephant random walk

- (Pure) Dynamic random walk.
  - Let (E, Σ, μ) be a complete probability space.
  - Let $T : E \to E$ be a one-to-one onto map such that T and T^{-1} are both measurable. Assume also that T is measure preserving, i.e.,

$$\mu(T^{-1}(A)) = \mu(A), \ \forall \ A \in \Sigma.$$

Dynamic elephant random walk

- (Pure) Dynamic random walk.
  - Let (E, Σ, μ) be a complete probability space.
  - Let $T : E \to E$ be a one-to-one onto map such that T and T^{-1} are both measurable. Assume also that T is measure preserving, i.e.,

$$\mu(T^{-1}(A)) = \mu(A), \ \forall \ A \in \Sigma.$$

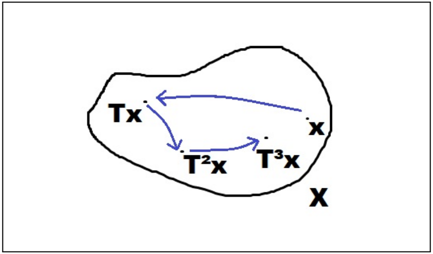
  - We call (E, Σ, μ, T) a dynamical system.

DRW

- Let $x \in E$. The orbit of x is given by $(x, Tx, T^2x, T^3x, \ldots)$ where $T^{n+1}x = T(T^n x)$.

DRW

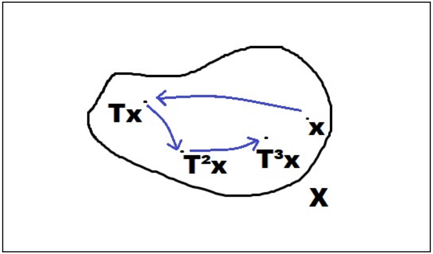
- Let $x \in E$. The orbit of x is given by $(x, Tx, T^2x, T^3x, \ldots)$ where $T^{n+1}x = T(T^nx)$.



- Let $f : E \to \mathbb{R}$ be a measurable map.

DRW

- Let $x \in E$. The orbit of x is given by $(x, Tx, T^2x, T^3x, \dots)$ where $T^{n+1}x = T(T^nx)$.



- Let $f : E \to \mathbb{R}$ be a measurable map.
- $f(x), f(T(x)), f(T^2(x)), \dots$ may represent some measurement made on the system.

DRW

- Let (E, Σ, μ, T) be a dynamical system and let $f : E \to [0, 1]$ be a measurable map.

DRW

- Let (E, Σ, μ, T) be a dynamical system and let $f : E \to [0, 1]$ be a measurable map.
- Let $(X_i)_i$ be $\mathbb{Z}$-valued random variables with law given by

$$\mathbb{P}[X_n = 1] = f(T^n x)$$

and

DRW

- Let (E, Σ, μ, T) be a dynamical system and let $f : E \to [0, 1]$ be a measurable map.
- Let $(X_i)_i$ be $\mathbb{Z}$-valued random variables with law given by

$$\mathbb{P}[X_n = 1] = f(T^n x)$$

and

$$\mathbb{P}[X_n = -1] = 1 - f(T^n x), \text{ for } n \geq 1.$$

DRW

- Let (E, Σ, μ, T) be a dynamical system and let $f : E \to [0,1]$ be a measurable map.
- Let $(X_i)_i$ be $\mathbb{Z}$-valued random variables with law given by

$$\mathbb{P}[X_n = 1] = f(T^n x)$$

and

$$\mathbb{P}[X_n = -1] = 1 - f(T^n x), \text{ for } n \geq 1.$$

- The sequence $(S_n)_{n \geq 0}$ given by $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ for any $n \geq 1$ is called a dynamic $\mathbb{Z}$- random walk.

Strong law for the DRW

- Using Kolmogorov criteria and Birkhoff ergodic theorem Guillotin-Plantar and Schott (2006) proved the following strong law of large number for DRW:

Strong law for the DRW

- Using Kolmogorov criteria and Birkhoff ergodic theorem Guillotin-Plantar and Schott (2006) proved the following strong law of large number for DRW:
- For μ-almost every $x \in E$

$$\frac{S_n}{n} \to 2\mathbb{E}[f|\mathcal{I}] - 1 \quad \mathbb{P} - q.c.,$$

Strong law for the DRW

- Using Kolmogorov criteria and Birkhoff ergodic theorem Guillotin-Plantar and Schott (2006) proved the following strong law of large number for DRW:
- For μ-almost every $x \in E$

$$\frac{S_n}{n} \to 2\mathbb{E}[f|\mathcal{I}] - 1 \quad \mathbb{P} - q.c.,$$

where $\mathcal{I}$ denotes the σ-algebra of T-invariant sets, i.e. $\mu(T^{-1}(A)\Delta A) = 0$.

Dynamic Random Elephant

About the ERW and the DRW. Remember that P_E is the increments law for the ERW and P_D is the increments law for the DRW.

- $P_E[X_n = \eta | X_1, \ldots, X_{n-1}] = \frac{1}{2n} \sum_{k=1}^{n} [1 + (2p - 1) X_k \eta]$ for $n \geq 1$;
- $P_E[X_1 = \eta] = \frac{1}{2}[1 + (2q - 1)\eta]$;
- $P_D[X_n = \eta] = \frac{1}{2}[1 + (2f(T^n x) - 1)\eta]$.
  Here, $\eta \in \{-1, 1\}$.

Dynamic Random Elephant

Modelo DRE

Let $g : \mathbb{R} \times \mathbb{N} \to [0, 1]$.

We say that the random walk $S_n = X_1 + \cdots + X_n$ is a DRE if

- $P[X_1 = \eta] = g(\alpha, 1)P_E[X_1 = \eta] + (1 - g(\alpha, 1))P_D[X_1 = \eta]$;
- $P[X_n = \eta] = g(\alpha, n)P_E[X_n = \eta | X_1, \ldots, X_n]$
  $+ (1 - g(\alpha, n))P_D[X_n = \eta]$

Results

First results

- $\mathbb{E}[X_{n+1}|X_1,\ldots,X_n] = \frac{\alpha_{n+1}(2p-1)}{n} S_n + (1-\alpha_{n+1})(2f(T^{n+1}x)-1)$;
  Set $a_n = \prod_{k=1}^{n-1}\left(1+\frac{g(\alpha,k+1)(2p-1)}{k}\right)$.
- $\mathbb{E}[S_n] = a_n\left(g(\alpha,1)(2q-1) + \sum_{k=1}^{n}\frac{(1-g(\alpha,k))(2\mathbb{E}[f(T^k x)]-1)}{a_k}\right)$

Results

Definition

We say that the DRE satisfies the **strong property** if any of the following statements hold

- $p = 1$ e $\lim_{n\to\infty} g(\alpha, n) = \delta \in [0, 1)$;
- $p \neq 1$ e $\lim_{n\to\infty} g(\alpha, n) = \delta \in [0, 1]$.

Strong law of large numbers

If the DRE satisfies the strong property, then

$$\lim_{n\to\infty} \left| \frac{S_n}{n} - \frac{\mathbb{E}[S_n]}{n} \right| = 0 \quad \text{a.s.} \tag{5}$$

Results

Lema

If the DRE satisfies the strong property, then

$$\lim_{n\to\infty} \frac{S_n}{n} = \frac{(1-\ell(\alpha))}{1-(2p-1)\ell(\alpha)}(2\mathbb{E}[f|\mathcal{I}]-1), \tag{6}$$

where $\ell(\alpha) = \lim_{n\to\infty} g(\alpha, n)$ and $L = \lim_{n\to\infty} f(T^n x)$.

Results

Set $A_n^2 = \sum_{k=1}^{n} \frac{1}{a_n^2}$.

Central Limit Theorem

Assume that the strong property holds, that $p \geq 1/2$ or, if $p < 1/2$, $\lim_{n\to\infty} g(\alpha, n) \leq \frac{1}{2-4p}$. Then

$$\frac{S_n - \mathbb{E}[S_n]}{a_n A_n} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \lambda), \tag{7}$$

where $\lambda = \lim_{n\to\infty} 1 - \left(\frac{S_n}{n}\right)^2$.

Results

Almost Sure Convergence Theorem

Assume that the strong property holds, that $p < 1/2$ and that $\lim_{n\to\infty} g(\alpha, n) > \frac{1}{2-4p}$. Then,

$$\frac{S_n - \mathbb{E}[S_n]}{a_n} \xrightarrow{a.s.} M \tag{8}$$

where M is a non–degenerate random variable with zero mean,

Two repelling rw. Stochastic approximation: An introduction

Two repelling rw. Stochastic approximation: An introduction

- Consider a coffee machine which can be working or not. Let y_n be the number of days the machine has been working and let $x_n = y_n/n$

Two repelling rw. Stochastic approximation: An introduction

- Consider a coffee machine which can be working or not. Let y_n be the number of days the machine has been working and let $x_n = y_n/n$
- Assume that the conditional probability that in the $n+1$ day the coffee machine will be working given the past up to the day n is just a function of $x_n, f(x_n)$. Then,

Two repelling rw. Stochastic approximation: An introduction

- Consider a coffee machine which can be working or not. Let y_n be the number of days the machine has been working and let $x_n = y_n/n$
- Assume that the conditional probability that in the $n+1$ day the coffee machine will be working given the past up to the day n is just a function of $x_n, f(x_n)$. Then,
- $y_{n+1} = y_n + z_{n+1}$ where

$$z_{n+1} = 1\{ \text{ coffee machine is working by day } n+1\}.$$

Then,

Stochastic approximation

- $x_{n+1} = x_n + \frac{1}{n+1}\left(z_{n+1} - x_n\right)$ with $x_0 = 0$.

Stochastic approximation

- $x_{n+1} = x_n + \frac{1}{n+1}(z_{n+1} - x_n)$ with $x_0 = 0$. Thus,

$$x_{n+1} = x_n + \frac{1}{n+1}(f(x_n) - x_n) + \frac{1}{n+1}(z_{n+1} - f(x_n)).$$

Stochastic approximation

- $x_{n+1} = x_n + \frac{1}{n+1}(z_{n+1} - x_n)$ with $x_0 = 0$. Thus,

$$x_{n+1} = x_n + \frac{1}{n+1}(f(x_n) - x_n) + \frac{1}{n+1}(z_{n+1} - f(x_n)).$$

- If $D_n := z_n - f(x_{n-1})$, then its mean is 0, its conditional expectation given z_n is zero and $\{D_n\}$ is a martingale difference sequence (A noise = uncorrelated with the past).

Stochastic approximation

- This equation may be thought as a noisy discretization for the ODE

Stochastic approximation

- This equation may be thought as a noisy discretization for the ODE
$$\dot{x}(t) = f(x(t)) - x(t)$$
for $t > 0$.

Two repelling random walks

- Consider two interacting random walks on $\mathbb{Z}$.

Two repelling random walks

- Consider two interacting random walks on $\mathbb{Z}$.
- The transition probability of one walk in one direction decreases exponentially with the number of transitions of the other walk in that direction.

Two repelling random walks

- The joint process may thus be seen as two random walks reinforced to repel each other.

Two repelling random walks

- The joint process may thus be seen as two random walks reinforced to repel each other.
- The strength of the repulsion is further modulated in our model by a parameter $\beta \geq 0$.
- We study the recurrence and transience of this random walk in terms of this parameter.

The model

- Consider two repelling random walks $\{S_n^i;\ i = 1, 2, n \geq 0\}$ taking values on $\mathbb{Z}$.

The model

- Consider two repelling random walks $\{S_n^i;\ i = 1, 2, n \geq 0\}$ taking values on $\mathbb{Z}$.
- The repulsion is determined by the full previous history of the joint process.

The model

- Consider two repelling random walks $\{S_n^i;\ i = 1, 2, n \geq 0\}$ taking values on $\mathbb{Z}$.
- The repulsion is determined by the full previous history of the joint process.
- Let $\mathcal{F}_n = \sigma(\{S_k^1, S_k^2 : 0 \leq k \leq n\})$.

The model

- The transition probability for each process is defined as

$$\mathbb{P}\big(S^i_{n+1} = S^i_n + 1 \,\big|\, \mathcal{F}_n\big) = \psi\big((S^j_n - S^j_0)/n\big) = 1 - \mathbb{P}\big(S^i_{n+1} = S^i_n - 1 \,\big|\, \mathcal{F}_n\big), \tag{9}$$

with $i = 1, 2$, $j = 3 - i$, $n \geq n_0$.

The model

- The transition probability for each process is defined as

$$\mathbb{P}\big(S^i_{n+1} = S^i_n + 1 \,\big|\, \mathcal{F}_n\big) = \psi\big((S^j_n - S^j_0)/n\big) = 1 - \mathbb{P}\big(S^i_{n+1} = S^i_n - 1 \,\big|\, \mathcal{F}_n\big), \tag{9}$$

with $i = 1, 2$, $j = 3 - i$, $n \geq n_0$.

- $\psi : [-1, 1] \to [0, 1]$, defined by

$$\psi(y) = \frac{1}{1 + \exp(\beta y)}, \quad \beta \geq 0. \tag{10}$$

The model

- The transition probability for each process is defined as

$$\mathbb{P}\big(S^i_{n+1} = S^i_n + 1 \,\big|\, \mathcal{F}_n\big) = \psi\big((S^j_n - S^j_0)/n\big) = 1 - \mathbb{P}\big(S^i_{n+1} = S^i_n - 1 \,\big|\, \mathcal{F}_n\big), \tag{9}$$

  with $i = 1, 2,\ j = 3 - i,\ n \geq n_0$.
- $\psi : [-1, 1] \to [0, 1]$, defined by

$$\psi(y) = \frac{1}{1 + \exp(\beta y)}, \quad \beta \geq 0. \tag{10}$$

- When $\beta = 0$, then $\psi(y) = \frac{1}{2}$ for all $y \in [-1, 1]$ and both S^1_n and S^2_n form two independent simple random walks on $\mathbb{Z}$.

Main results

We regard a walk S_n^i as recurrent (transient) if every vertex of $\mathbb{Z}$ is visited by S_n^i infinitely (finitely) many times almost surely.

Main results

We regard a walk S_n^i as recurrent (transient) if every vertex of $\mathbb{Z}$ is visited by S_n^i infinitely (finitely) many times almost surely.

Theorem

If $\beta > 2$, both random walks S_n^1 and S_n^2 are transient and

$$\lim_{n\to\infty} S_n^1 = -\lim_{n\to\infty} S_n^2 = \pm\infty \quad \text{a.s.}$$

Main results

We regard a walk S_n^i as recurrent (transient) if every vertex of $\mathbb{Z}$ is visited by S_n^i infinitely (finitely) many times almost surely.

Theorem

If $\beta > 2$, both random walks S_n^1 and S_n^2 are transient and

$$\lim_{n\to\infty} S_n^1 = -\lim_{n\to\infty} S_n^2 = \pm\infty \quad \text{a.s.}$$

Theorem

If $\beta \in [0,1]$, then both S_n^1 and S_n^2 are recurrent.

A dynamical system approach

For $n \geq 0$, $i = 1, 2$, set

$$\xi(n) = \left(\xi_l^1(n), \xi_r^1(n), \xi_l^2(n), \xi_r^2(n)\right), \quad \xi_l^i(n) = \mathbf{1}_{\{S_{n+1}^i - S_n^i = -1\}}, \quad (11)$$

A dynamical system approach

For $n \geq 0,\ i = 1, 2$, set

$$\xi(n) = \left(\xi_l^1(n), \xi_r^1(n), \xi_l^2(n), \xi_r^2(n)\right), \quad \xi_l^i(n) = \mathbf{1}_{\{S_{n+1}^i - S_n^i = -1\}}, \quad (11)$$

and

$$\xi_r^i(n) = \mathbf{1}_{\{S_{n+1}^i - S_n^i = 1\}}.$$

A dynamical system approach

For $n \geq 0$, $i = 1, 2$, set

$$\xi(n) = \left(\xi_l^1(n), \xi_r^1(n), \xi_l^2(n), \xi_r^2(n)\right), \quad \xi_l^i(n) = \mathbf{1}_{\{S_{n+1}^i - S_n^i = -1\}}, \quad (11)$$

and

$$\xi_r^i(n) = \mathbf{1}_{\{S_{n+1}^i - S_n^i = 1\}}.$$

Also, let

$$X_l^i(n) = \frac{1}{n} \sum_{k=0}^{n-1} \xi_l^i(k), \qquad X_r^i(n) = \frac{1}{n} \sum_{k=0}^{n-1} \xi_r^i(k),$$

A dynamical system approach

For $n \geq 0$, $i = 1, 2$, set

$$\xi(n) = \left(\xi_l^1(n), \xi_r^1(n), \xi_l^2(n), \xi_r^2(n)\right), \quad \xi_l^i(n) = \mathbf{1}_{\{S_{n+1}^i - S_n^i = -1\}}, \quad (11)$$

and

$$\xi_r^i(n) = \mathbf{1}_{\{S_{n+1}^i - S_n^i = 1\}}.$$

Also, let

$$X_l^i(n) = \frac{1}{n}\sum_{k=0}^{n-1} \xi_l^i(k), \qquad X_r^i(n) = \frac{1}{n}\sum_{k=0}^{n-1} \xi_r^i(k),$$

Denote by $X = \{X(n)\}_{n \geq 0}$ the process determined by

$$X(n) = (X_l^1(n), X_r^1(n), X_l^2(n), X_r^2(n))$$

A dynamical system approach

The process X takes values on the set

$$\mathcal{D} = \triangle \times \triangle$$

where

$$\triangle = \{x \in \mathbb{R}^2 \mid x_v \geq 0, \sum_v x_v = 1\}.$$

A dynamical system approach

The process X takes values on the set

$$\mathcal{D} = \triangle \times \triangle$$

where

$$\triangle = \{x \in \mathbb{R}^2 \mid x_v \geq 0, \sum_v x_v = 1\}.$$

Some notation.

A dynamical system approach

The process X takes values on the set

$$\mathcal{D} = \triangle \times \triangle$$

where

$$\triangle = \{x \in \mathbb{R}^2 \mid x_v \geq 0, \sum_v x_v = 1\}.$$

Some notation. Now, let $\pi : \mathcal{D} \to \mathcal{D}$ be the map

$$x \mapsto \pi(x) = \left(\pi_l^1(x), \pi_r^1(x), \pi_l^2(x), \pi_r^2(x)\right) \tag{12}$$

where for $i = 1, 2$ and $v = l, r$,

$$\pi_v^i(x) = \psi(2x_v^j - 1), \qquad j = 3 - i. \tag{13}$$

A dynamical system approach

Lemma

The process $X = \{X(n)\}_{n\geq 0}$ *satisfies the following recursion*

$$X(n+1) - X(n) = \gamma_n(F(X(n)) + U_n) \tag{14}$$

where

$$\gamma_n = \frac{1}{n+1} \tag{15}$$

and $F : \mathcal{D} \to T\mathcal{D}$ *is the vector field* $F = (F_l^1, F_r^1, F_l^2, F_r^2)$ *defined by*

$$F(X(n)) = -X(n) + \pi(X(n)). \tag{16}$$

A dynamical system approach

- A discrete-time process whose increments are recursively computed according to (14) is known as a stochastic approximation.

A dynamical system approach

- A discrete-time process whose increments are recursively computed according to (14) is known as a stochastic approximation.
- Provided the random term U_n can be damped by γ_n, (14) may be thought as a Cauchy-Euler approximation scheme,

$$x(n+1) - x(n) = \gamma_n F(x(n)),$$

for the numerical solution of the autonomous ODE

A dynamical system approach

- A discrete-time process whose increments are recursively computed according to (14) is known as a stochastic approximation.
- Provided the random term U_n can be damped by γ_n, (14) may be thought as a Cauchy-Euler approximation scheme,

$$x(n+1) - x(n) = \gamma_n F(x(n)),$$

for the numerical solution of the autonomous ODE

$$\dot{x} = F(x).$$

A dynamical system approach

- A natural approach to determine the limit behaviour of the process X consists in studying the asymptotic properties of the related ODE.

A dynamical system approach

- A natural approach to determine the limit behaviour of the process X consists in studying the asymptotic properties of the related ODE.
- This heuristic, known as the ODE method, has been rather effective while studying various reinforced stochastic processes.

Equilibria

- A point $x \in \mathcal{D}$ is an equilibrium of F if $F(x) = 0$.

Equilibria

- A point $x \in \mathcal{D}$ is an equilibrium of F if $F(x) = 0$.
- For any point $x \in \mathcal{D}$, let $\mathcal{J}_F(x)$ be the Jacobian matrix of the vector field F at x and let $\sigma(\mathcal{J}_F(x))$ be the set of its eigenvalues.

Equilibria

- The equilibrium x is hyperbolic if all the eigenvalues of $\sigma(\mathcal{J}_F(x))$ have non-zero real parts.

Equilibria

- The equilibrium x is hyperbolic if all the eigenvalues of $\sigma(\mathcal{J}_F(x))$ have non-zero real parts.
- The hyperbolic equilibrium x is linearly stable if $\sigma(\mathcal{J}_F(x))$ contains only eigenvalues with negative real parts; otherwise x is said to be linearly unstable.

Convergence to equilibria

Theorem

Assume that $X = (X(n))_n$ *be a process satisfying our recursion equation. Then, for any* $\beta \geq 0, \beta \neq 2$*, the process* X *converges a.s. to an equilibrium point of our vector field.*

Convergence to equilibria

Lemma

For $\beta \in [0,2]$, the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the only equilibrium for the vector field F. For any $\beta > 2$, the field has three equilibria,

Convergence to equilibria

Lemma

For $\beta \in [0,2]$, the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the only equilibrium for the vector field F. For any $\beta > 2$, the field has three equilibria,

$$\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \quad (w, 1-w, 1-w, w) \quad \textit{and} \quad (1-w, w, w, 1-w), \tag{17}$$

where $w \in (0, \frac{1}{2})$ is uniquely determined by β.

Convergence to equilibria

Lemma

For $\beta \in [0, 2]$, the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the only equilibrium for the vector field F. For any $\beta > 2$, the field has three equilibria,

$$\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \quad (w, 1-w, 1-w, w) \quad \text{and} \quad (1-w, w, w, 1-w), \tag{17}$$

where $w \in (0, \frac{1}{2})$ is uniquely determined by β. The equilibrium $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is linearly stable for $\beta \in [0, 2)$ and linearly unstable for $\beta > 2$.

Convergence to equilibria

Lemma

For $\beta \in [0,2]$, *the point* $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ *is the only equilibrium for the vector field* F. *For any* $\beta > 2$, *the field has three equilibria,*

$$\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \quad (w, 1-w, 1-w, w) \quad \text{and} \quad (1-w, w, w, 1-w), \tag{17}$$

where $w \in (0, \frac{1}{2})$ *is uniquely determined by* β. *The equilibrium* $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ *is linearly stable for* $\beta \in [0,2)$ *and linearly unstable for* $\beta > 2$. *The equilibria* $(w, 1-w, 1-w, w)$ *and* $(1-w, w, w, 1-w)$ *are linearly stable for* $\beta > 2$.

Non-convergence to the unstable equilibrium

Lemma

Let $X = \{X(n)\}_{n \geq 0}$ *be a process satisfying an stochastic approximation recursion. Then, if* $\beta > 2$,

$$\mathbb{P}\Big(\lim_{n\to\infty} X(n) = \big(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big)\Big) = 0.$$

Almost sure convergence of the proportions

Lemma

There is a unique point $x \in [0,1]$, depending on β, such that,

$$\lim_{n\to\infty} \frac{1}{n}\Big(S_n^1 - S_{n_0}^1,\ S_n^2 - S_{n_0}^2\Big) \ \in\ \Big\{(x,-x),\ (-x,x)\Big\} \qquad \text{a.s.}$$

In addition, if $0 \le \beta \le 2$, then $x = 0$, and if $\beta > 2$, then $0 < x < 1$.

Proof of transience

- It follows from the previous lemma that $\left(S_n^1/n, S_n^2/n\right)$ converges a.s. to $(x, -x)$ or to $(-x, x)$ where $x > 0$.

Proof of transience

- It follows from the previous lemma that $\left(S_n^1/n, S_n^2/n\right)$ converges a.s. to $(x, -x)$ or to $(-x, x)$ where $x > 0$.
- Since $S_n^i = S_n^i/n \times n$, the proof is complete after making $n \to +\infty$.

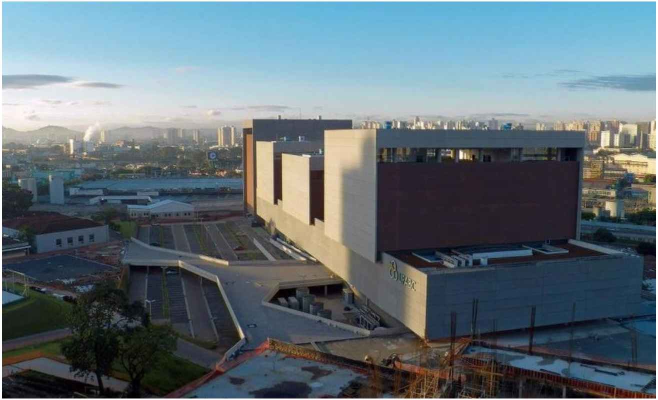


Figura: UFABC – Campus S.A.

Thanks.

Bibliography

- Coletti, Cristian F., Renato Gava, and Gunter M. Schutz. "Central limit theorem and related results for the elephant random walk." Journal of mathematical physics 58, no. 5 (2017).
- Coletti, Cristian F., Renato Gava, and Gunter M. Schutz. "A strong invariance principle for the elephant random walk." Journal of Statistical Mechanics: Theory and Experiment 2017, no. 12 (2017): 123207.
- Prado, Fernando PA, Cristian F. Coletti, and Rafael A. Rosales. "Two repelling random walks on Z." Stochastic Processes and their Applications 160 (2023): 72-88.

Bibliography

- Coletti, Cristian F., and Ioannis Papageorgiou. "Asymptotic analysis of the elephant random walk." Journal of Statistical Mechanics: Theory and Experiment 2021, no. 1 (2021): 013205.
- Coletti, Cristian F., Lucas R. de Lima, Renato J. Gava, and Denis A. Luiz. "Limit theorems for a random walk with memory perturbed by a dynamical system." Journal of Mathematical Physics 61, no. 10 (2020).
- Coletti, Cristian F., Renato J. Gava, and Lucas R. de Lima. "Limit theorems for a minimal random walk model." Journal of Statistical Mechanics: Theory and Experiment 2019, no. 8 (2019): 083206.