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THE RADICAL OF AN ALTERNATIVE RING

BY M. F. SMILEY

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Introduction

In this paper we shall show that N. Jacobson's definition of the radical of an (associative) ring [J1, B] applies to alternative rings [Z1, M], and we shall develop some of the elementary properties of this radical. The radical of an alternative ring was first discussed by M. Zorn [Z2] under certain finiteness assumptions. Dubisch and Perlis [D - P] have more recently studied the radical of an alternative algebra (of finite order). We make no finiteness assumptions. We do, however, prove that the chain conditions employed by Zorn [Z2, (4.2.1)–(4.2.3)] ensure that the radical defined by him coincides with that defined in this paper. Our discussion applies equally well to algebras of possibly infinite order [J1, §6] so that our results essentially contain some of the results of Dubisch and Perlis.

We have been unable to discover the relation between the radical and maximal right ideals, nor have we developed any parallel for Jacobson's structure theory of associative rings. The enlarged radical of Brown and McCoy [B-McC] will yield a type of structure theory for arbitrary non-associative rings, but because of the generality involved we prefer to leave a discussion of this interesting fact to a subsequent publication.

1. Preliminaries

We shall assume throughout this paper that A is an alternative ring, that is, that every two elements of A generate an associative subring. Our principal computations will involve the *associator* $(a, b, c) = a(bc) - (ab)c$ of three elements $a, b, c \in A$. Zorn [Z2, (1.6)] has established the identities

$$(1) \quad (ab, b, c) = b(a, b, c), \quad (ba, b, c) = (a, b, c)b,$$

which will be fundamental in our computations. We shall not need any of the other identities developed in [Z2] except the basic fact that the associator (a, b, c) is an alternating function of its arguments in the sense that it changes sign on the interchange of two of its arguments and is zero if two of its arguments are equal.¹

The following theorem was proved by R. Moufang [M] for the case of alternative *division* rings.

THEOREM 1. GENERALIZED THEOREM OF ARTIN. *If $(a, b, c) = 0$, then the subring of A generated by a, b, c is associative.*

A proof of this theorem, modelled on that of Miss Moufang, is given in an appendix to this paper. Theorem 1 is much more general than needed to justify

¹ This statement is not intended to apply to the Appendix.

the arguments we shall make and the reader who prefers to omit its proof should encounter no difficulty in using (1) to supply the missing details.

As a matter of notation, we shall denote the right ideal generated by an element $a \in A$ by $r(a)$. We note that $r(a)$ consists of the totality of elements of A of the form $ai + \sum aU_j$, where i is an integer and each U_j is a product of a finite number of right multiplications $R_x : aR_x = ax$.

2. The quasi-addition of Perlis

This section will be devoted to a cursory examination of the quasi-addition of Perlis [P] in an alternative ring. We are interested in deriving just those properties of this operation which will be useful in our subsequent definition of the radical, and we do not attempt a systematic development. We feel, however, that such a development might be quite interesting for its own sake.

The *quasi-sum* aob of two elements of A is defined to be $aob = a + b + ab$, and we call the operation aob *quasi-addition*. The zero of A is a unit for quasi-addition and we say that $a \in A$ is *right quasi-regular* in case there is an element $b \in A$ for which $aob = 0$. When this is true we call b a *right quasi-inverse* of a .

Our first lemma lists some identities which are easy to verify and which are independent of our basic assumption that A is alternative.

LEMMA 1. *If $a, b, c \in A$, then*

$$\begin{aligned} ao(boc) - (aob)oc &= (a, b, c), \\ (a + b)oc &= (aob) + (boc) - c, \\ c(aob) - (coa)b &= ca - ab + (c, a, b). \end{aligned}$$

LEMMA 2. *If $aob = 0$, then $(a, b, c) = 0$ for every $c \in A$.*

PROOF. Set $\alpha = (a, b, c)$ and compute $0 = (aob, b, c) = \alpha + (ab, b, c) = \alpha + b$, using (1). In like manner, $0 = (a, aob, c) = \alpha + (a, ab, c) = \alpha + \alpha a$. But then $(b, \alpha, a) = b(\alpha a) - (ba)\alpha = -b\alpha + \alpha a = 0$. Hence

$$0 = \alpha + \alpha a + (\alpha + \alpha a)b = \alpha + \alpha(a + b + ab) = \alpha,$$

and the proof is complete.

REMARK. A similar argument shows that if A has a unit element 1, and if $ab = 1$, then $(a, b, c) = 0$ for every $c \in A$ [Cf Z2, (2.4)]. Thus if also $xa = 1$, then $x = b$.

COROLLARY 1. *If $aob = coa = 0$, then $ab = ca$ and $b = c$.*

COROLLARY 2. *If $a + ab = 0$ and b is right quasi-regular, then $a = 0$.*

PROOF. If $b + c + bc = 0$, then $ab + ac + (ab)c = ab = -a = 0$.

COROLLARY 3. *If $e \in A$ is a nonzero idempotent, then $-e$ is not right quasi-regular.*

LEMMA 3. *If ab is right quasi-regular, then ba is right quasi-regular.*

PROOF. If $(ab)oc = 0$, we shall show that $\alpha = (a, b, c) = 0$. First $b\alpha = b(a, b, c) = (ab, b, c) = 0$ by Lemma 2. Likewise, $\alpha a = (a, b, c)a = (a, ab, c) = 0$. Consequently, $(b, \alpha, a) = 0$. But, since $ab + c + (ab)c = 0$, we find that

$$\begin{aligned} \alpha &= (a, b, - (ab)c) = (a, b, \alpha - a(bc)) = - (a, b, a(bc)) \\ &= - ((a, b, bc)a) = - \alpha ba. \end{aligned}$$

Now, the generalized theorem of Artin gives $\alpha b + \alpha bab = 0$, and Corollary 2 of Lemma 2 yields $\alpha b = 0$ from which $\alpha = 0$ follows. The generalized theorem of Artin may be employed again to verify the identity [Cf. K, p. 154]

$$(ba)o(-ba - bca) = -b((ab)oc)a.$$

The proof is complete.

3. Definition of the radical

Our main goal in this section is to define the radical and to show that it is an ideal.

Following Jacobson [J1] we define the right radical $R_r(A)$ of A as the totality of elements $a \in A$ for which each element of $r(a)$ is right quasi-regular. The left radical $R_l(A)$ is defined in like manner.

LEMMA 4. *If $a \in R_r(A)$ and b is right quasi-regular, then $a + b$ is right quasi-regular.*

PROOF. We have $boc = 0$ for some $c \in A$. Then $a + ac \in r(a)$, so that $(a + ac)od = 0$ for some $d \in A$. We shall show that $(a + b)o(cod) = 0$. First, by Lemma 1, we find that

$$\begin{aligned} (a + b)o(cod) &= ((a + b)oc)od + (a + b, c, d) \\ &= (a + ac)od + (a + b, c, d). \end{aligned}$$

Now, by Lemma 2, we obtain $(a + b)o(cod) = (a, c, d)$. Again, Lemma 2 gives $(a + ac, c, d) = 0$. Setting $\alpha = (a, c, d)$ and applying (1) yields $\alpha + c\alpha = 0$, from which $\alpha = 0$ follows from the dual of Corollary 2 of Lemma 2. The proof is complete.

LEMMA 5. *The right radical $R_r(A)$ of A is a right ideal.*

PROOF. If $a \in R_r(A)$ and $c \in A$, it is clear that $-a$ and ac are each in $R_r(A)$. If also $b \in R_r(A)$, then $r(a + b)$ consists of all elements of the form

$$(a + b)i + \sum (a + b)U_j = ai + \sum aU_j + bi + \sum bU_j$$

where i is an integer and each U_j is the product of a finite number of right multiplications of A . Each such element is right quasi-regular by Lemma 4, so that $(a + b) \in R_r(A)$. The proof is complete.

LEMMA 6. *If I is a right ideal and $a \in I$ while $b, c \in A$, then $(a, b, c) \in I$.*

PROOF. This is clear since $(a, b, c) = a(bc) - (ab)c$.

LEMMA 7. *The right radical $R_r(A)$ is a left ideal.*

PROOF. Let $a \in R_r(A)$ and $b \in A$. Then $r(ba)$ consists of all elements of the form $(ba)i + \sum (ba)U_j$, where i is an integer and each U_j is the product of a finite number of right multiplications of A . We shall prove by induction on the number of factors of U_j that $(ba)U_j = bu + v$ for some elements $u, v \in R_r(A)$.

If $U_j = R_x$, then $(ba)G_j = (ba)x = b(ax) - (b, a, x)$, and $(b, a, x) \in R_r(A)$ by Lemma 6. To complete the induction, note that

$$(bs + t)R_y = (bs)y + ty = b(sy) - (b, s, y) + ty,$$

from which $(bs + t)R_y = bu + v$ with $u, v \in R_r(A)$ follows by Lemmas 5 and 6 provided that $s, t \in R_r(A)$. Thus $r(ba)$ consists of all elements $b(w + ai) + z$, with i an integer and $w, z \in R_r(A)$. Each such element is right quasi-regular by Lemmas 3 and 4. Hence $(ba) \in R_r(A)$ and $R_r(A)$ is a left ideal.

LEMMA 8. *The right radical $R_r(A)$ coincides with the left radical $R_l(A)$.*

PROOF. Let $a \in R_r(A)$ and let $b \in A$ be an element in the left ideal generated by a . Then $b \in R_r(A)$ by Lemma 7. Consequently $boc = b + c + bc = 0$ for some $c \in A$ and $c = -b - bc \in R_r(A)$. It follows by Corollary 1 of Lemma 2 that $cob = 0$, that is, that b is left quasi-regular. Thus $R_r(A) \subseteq R_l(A)$. Dually, $R_l(A) \subseteq R_r(A)$. The proof is complete.

We define the radical $R(A)$ of A as $R(A) = R_r(A) = R_l(A)$.

THEOREM 2. *The radical $R(A)$ of an alternative ring A is a two-sided ideal of A .*

PROOF. This is clear.

4. Elementary properties of the radical

We shall list in this section some immediate consequences of our definition of the radical of A . We shall give no proofs because Lemma 2 shows that the proofs of the corresponding statements given by Jacobson for the associative case [J1] are valid here. It may be observed, however, that use of Corollary 2 of Lemma 2 will shorten these proofs slightly.

THEOREM 3. *Let A^* be an alternative ring with unit element 1 containing A and such that $A^* = A + (1)$, where (1) denotes the subring of A^* generated by 1. Then $R(A) = R(A^*) \cap A$. If also $A \cap (1) = 0$ and (1) is isomorphic to the ring of integers, then $R(A) = R(A^*)$.*

THEOREM 4. *We have $R(A - R(A)) = 0$.*

THEOREM 5. *If I is a right (left) ideal of A and if each element of I is nilpotent, then $I \subseteq R(A)$.*

THEOREM 6. *If B is a subset of $R(A)$ which is closed under multiplication and if $b \in B$, then for every positive integer n either $b^{n-1}B > b^nB$ or $b^n = 0$.*

THEOREM 7. *If A is regular in the sense of von Neumann, then $R(A) = 0$.*

REMARK. That $R(A)$ contains no nonzero idempotents may be seen by Corollary 3 of Lemma 2 or as a corollary of Theorem 6.

5. The radical of a hypercomplex alternative ring

In his discussion of the radical of an alternative ring, Zorn made use of certain chain conditions [Z2, (4.2.1)–(4.2.3)]. In this section we shall show that when these conditions hold the radical $R(A)$ coincides with the radical defined by Zorn. It will not be necessary for us to state Zorn's chain conditions explicitly for we shall actually use the following consequence of them: *If $a \in A$, then either a is nilpotent or there is an element $b \in A$ such that ab is a nonzero idempotent of A [Z2, (3.2)].*

THEOREM 8. *If the chain conditions of Zorn hold in A , then $R(A)$ consists of the set of all properly nilpotent elements of A .*

PROOF. Let $Z(A)$ denote the set of all properly nilpotent elements of A . Zorn showed that $Z(A)$ is an ideal of A . Thus $Z(A) \leq R(A)$ by Theorem 5. On the other hand, let $a \in R(A)$ and suppose that some multiple ax of a is not nilpotent. Then there is an element $b \in A$ for which $(ax)b$ is a nonzero idempotent of A . But $-(ax)b \in R(A)$, since $R(A)$ is an ideal of A . This is impossible by the Remark which closes Section 4. Thus every multiple ax of a is nilpotent and $a \in Z(A)$. Consequently $R(A) \leq Z(A)$ and the proof is complete.

6. Miscellaneous results

This section contains some observations which we have postponed in order to present our principal conclusions as rapidly as possible.

Our first theorem concerns the extension of R. Baer's definition of right quasi-regularity [J1] to the alternative case.

THEOREM 9. *If $a \in A$ then a is right quasi-regular if and only if every element of A has the form $ax + x$ for some $x \in A$.*

PROOF. If $a = ax + x$ for some $x \in A$, then $ao(-x) = 0$ and a is right quasi-regular. If $aob = 0$ and $c \in A$, then $(a + b + ab)c = 0$ and Lemma 2 yields $c = c + ac + bc + a(bc) = a(c + bc) + c + bc = ax + x$ for $x = c + bc \in A$.

Note, however, that if $a \in A$ is not right quasi-regular, we do not know whether the set of all elements of the form $ax + x$ for $x \in A$ is a right ideal. This is the difficulty encountered in studying the relation between $R(A)$ and the maximal right ideals of A .

LEMMA 9. *If A is commutative and $\alpha = (a, b, c)$ for $a, b, c \in A$, then $\alpha^3 = 0$.*

PROOF. If A is commutative, it is known that $3\alpha = 0$ and that $(u^3, v, w) = 0$ for every $u, v, w \in A$ [S]. Set $s = a(bc)$, $t = (ab)c$, and compute $\alpha^3 = (s - t)^3 = s^3 - 3s^2t + 3st^2 - t^3 = s^3 - t^3 - st(3(s - t)) = s^3 - t^3 - st(3\alpha) = 0$.

THEOREM 10. *If A is commutative and $R(A) = 0$, then A is an associative ring.*

PROOF. Let $a, b, c \in A$ and define $\alpha = (a, b, c)$. It is easy to verify that if $u \in r(\alpha)$, then $u^5 = 0$ so that $\alpha \in R(A)$ by Theorem 5. Since $R(A) = 0$, $\alpha = 0$ and A is an associative ring.

REMARK. Theorem 10 contains an observation of R. H. Bruck to the effect that a commutative alternative division ring is associative [S] as well as the corollaries of this observation which we stated in [S]. Theorem 10 also provides a simple proof of the fact that alternative rings which satisfy the condition $a^{n(a)} = a$ ($n(a)$ an integer greater than one) of Jacobson [J2, F-McC] are commutative and associative. The following simple example of a commutative alternative algebra A of order six over the prime field of three elements is due to Irving Kaplansky. The basis elements are x, y, z, u, v, w and the only nonzero products of two basis elements are $xy = u, yz = v, xv = w$, and $uz = -w$. Since every element of A is nilpotent, $R(A) = A$. It is clear that A is not associative since $(xy)z = uz = -w$, while $x(yz) = xv = w$.

APPENDIX

Proof of the generalized theorem of Artin

In addition to the identities used in the main body of our paper we shall require also the following one which holds in any non-associative ring [Z2, (1.2)].
 (2) $(ab, c, d) - (a, bc, d) + (a, b, cd) = a(b, c, d) + (a, b, c)d$. As a notational convenience we shall interpret the symbol a^0 in a product as a factor to be suppressed. A product all of whose factors are suppressed will quite arbitrarily be defined as zero.

We first prove two lemmas.

LEMMA 1. *If $(a, b, c) = 0$ and k, l, m are nonnegative integers, then $(a^k, b^l, c^m) = 0$.*

PROOF. It will clearly suffice to show that if k is a nonnegative integer, then

$$(3) \quad (a^k, b, c) = 0.$$

Using induction on k , we assume that (3) is valid for all nonnegative integers less than k . Then, if $k = 2p$, (2) yields $(a^k, b, c) = (a^p, a^p b, c)$, and (1) then gives $(a^p, a^p b, c) = (a^p, b, c)a^p = 0$. Again, if $k = 2p + 1$, then (2) gives $(a^k, b, c) = (a^p, a^{p+1}b, c) = (a^p, ab, c)a^p$ by (1). Using (2) again we find that $(ab, a^p, c) = (a, ba^p, c) - (a, b, a^p c)$. Successive applications of (1) show that $(a, ba^p, c) = (a, b, a^p c) = 0$ and the induction is complete.

LEMMA 2. *If $(a, b, c) = 0$ and l and m are nonnegative integers, then $(ab^l, b^m c) = a(b^{l+m}c)$.*

PROOF. For $l = 1$, the lemma follows from (1). Using (2), we find that $(ab^{l-1}, b, b^m c) - (a, b^l, b^m c) + (a, b^{l-1}, b^{m+1}c) = 0$, which, by (1), completes the induction.

We now attack the proof of the theorem. We write

$$P(3n) = a^{k_1} b^{l_1} c^{m_1} \dots a^{k_n} b^{l_n} c^{m_n},$$

where the exponents are nonnegative integers. We assume as our hypothesis of induction that every product of at most $3n$ factors of the form $a^{k_i}, b^{l_i}, c^{m_i}$ is uniquely defined irrespective of association. That our hypothesis is valid for $n = 1$ is the substance of Lemmas 1 and 2. We write $P(3n + 1) = P(3n)a^k$ for a nonnegative integer $k = k_{n+1}$. For each $s = 1, \dots, n$, there are three possibly different values obtained by associating the factors of $P(3n + 1)$. These are $(a^{k_1} \dots a^{k_s})(b^{l_s} \dots a^k)$, $(a^{k_1} \dots b^{l_s})(c^{m_s} \dots a^k)$, and $(a^{k_1} \dots c^{m_s})(a^{k_{s+1}} \dots a^k)$. To see that the first of these has the value $P(3n + 1)$, we compute

$$\begin{aligned} (a^{k_1} \dots a^{k_s}, b^{l_s} \dots c^{m_n}, a^k) &= (a^k, a^{k_1} \dots a^{k_s}, b^{l_s} \dots c^{m_n}) \\ &= a^k(a^{k_1}(b^{l_1} \dots c^{m_n}) - (a^{k+k_1}b^{l_1} \dots a^{k_s}b^{l_s} \dots c^{m_n})) = 0. \end{aligned}$$

A similar argument applies in the other two cases. If association occurs within one of the factors of $P(3n + 1)$, as

$$(a^{k_1} \dots a^{k_s} b^\mu)(b^r c^{m_s} \dots a^k)$$

with $\mu + \nu = l_s$, then Lemma 2 applies since

$$(a^{k_1} \dots a^{k_s}, b, c^{m_s} \dots a^k) = 0.$$

Now we write $P(3n + 2) = P(3n + 1)b^l$, where $l = l_{n+1}$ is a nonnegative integer. Again, for each $s = 1, \dots, n$, there are three different values to be obtained by associating the factors of $P(3n + 2)$. Two of these associations yield to the method of the previous paragraph, but the third one:

$$(a^{k_1} \dots c^{m_s}) (a^{k_{s+1}} \dots a^k b^l)$$

seems to require further discussion. We employ the device of Miss Moufang, setting $u = a^{k_1} \dots c^{m_s}$; $v = a^{k_{s+1}}$; $w = b^{l_{s+1}} \dots c^{m_n} a^k b^l$; $x = a^{k_1}$; and $y = b^{l_1} \dots c^{m_s}$. Then $xy = u$ and $(yx)w = y(xw)$ follow from the result of the previous paragraph. If we replace x by vx we find that $(vu)w = (vxy)w = (vx)(yw) = v(x(yw)) = v(xy)w = v(uw)$. It follows that $(uw)w = u(vw)$, as desired. Again Lemma 2 justifies association within a factor.

Finally we write $P(3n + 3) = P(3n + 2)c^m$, where $m = m_{n+1}$ is a nonnegative integer. Again for $s = 1, \dots, n$, there are three possibly different values to be obtained by associating the factors of $P(3n + 3)$. Again two of these yield to the method applied to $P(3n + 1)$, but the third one:

$$(a^{k_1} \dots a^{k_{s+1}}) (b^{l_{s+1}} \dots a^k b^l c^m)$$

requires further argument. We set $z = b^{l_{s+1}} \dots a^k b^l c^m$ and use the notation of the previous paragraph. Then $(yx)z = y(xz)$ follows from the result of the previous paragraph. If we replace x by vx we find that $(vu)z = (vxy)z = (vx)(yz) = v(x(yz)) = v((xy)z) = v(uz)$ and hence that $(uw)z = u(vz)$ as desired. A final application of Lemma 2 justifies association within the factors of

$$P(3n + 3)$$

and the induction is complete. Since every element in the subring of A which is generated by the elements a, b, c is a sum of terms of the form $P(3n)$, this completes the proof of the generalized theorem of Artin.

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