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ABSTRACT

Problem solving in mathematics incorporates abstract thinking skills, number logic skills, and a nearly endless list of other factors. One problem-solving task required in university-level mathematics courses is integration. This study investigated instructor presentation of a specific problem-solving strategy and whether it impacted performance on integration tasks of (n=95) students in college-level general calculus. Findings showed: (1) Instruction in a problem-solving strategy for integration tasks benefited those subjects receiving this instruction when it was paired with traditional instruction as opposed to those subjects who received traditional instruction only; (2) Females tended to benefit more from the treatment instruction than did males; and (3) Long-term retention of integration tasks was no better among those receiving the problem-solving strategy instruction. Appendices contain lesson plans for control and experimental groups, pre- and post-tests, the "integration" module, and statistical analyses. (MKR)

AN INVESTIGATION INTO A PROBLEM-SOLVING STRATEGY FOR INDEFINITE INTEGRATION AND ITS EFFECT ON TEST SCORES OF GENERAL CALCULUS STUDENTS

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INTRODUCTION

In recent years, representatives from business and education have expressed concern for the lack of problem-solving abilities among students. This presumed lack of a mathematics-based skill on the part of students at all levels of educational instruction is relatively easy to prove by simply glancing at the headlines and editorial pages contained within the popular press. Government agencies as well have noted the decline in national test scores for the past 20 years (National Commission on Excellence in Education, 1983).

In response to this and other similar education-based problems, the National Council of Teachers of Mathematics published its Curriculum and Evaluation Standards for School Mathematics (1989). Among the many topics contained within the mathematics-based curriculum, problem-solving is listed as one of the key elements needing emphasis and change.

A Rationale for the Study

Problem-solving in mathematics incorporates abstract thinking skills, number logic skills, and a nearly endless list of other factors. Generally, these skills may be divided into technical skills, *i.e.*, the mechanics of multiplication, drill-and-practice tasks, etc. and theoretical skills, *i.e.*, the higher order thought processes involved in not only solving a problem, but understanding why the problem is solved.

One problem-solving task required in university-level mathematics courses is integration. Schoenfeld (1978) stated that indefinite integration, a specific concept in a typical university-level calculus course, seems to cause students a great deal more difficulty than warranted by the topic. Integration, sometimes represented as the calculation of the area contained under a curve, is based on technical skills rather than theoretical knowledge, with the expectation that students should be able to calculate results when given ample instruction and practice. Test results show otherwise.

A Review of Previous Studies

Could test scores be improved if students were presented with a specific strategy for choosing a particular technique to solve indefinite integration problems? Previous research does not clearly define a specific outcome using general and/or specific problem-solving strategies. Schoenfeld's (1978) study with calculus students suggests that an instructor's presentation of a specific strategy can have a positive impact on test scores when assessment is conducted to determine whether a student has *"learned"* that strategy. There appear to be no previous studies dealing specifically with integration strategies in general calculus courses. Other research studies have failed to provide definitive results regarding the value of teaching specific strategies for mathematics instruction. This study incorporates a specific strategy for instruction within a general calculus course.

A Statement of the Research Problems

While reviewing the previous research conducted upon this topic, several questions arose that serve as the focus of this study. They include:

1. *Following presentation of a problem-solving strategy to solve specific types of integration problems, will mean test scores differ significantly between experimental and control groups in a course of general calculus?*
2. *Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will gender prove significant when mean test scores for students given the experimental treatment are compared with those in the control group?*
3. *Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will there be a significant difference in long-term retention mean test scores for the control group when compared to the experimental group?*

Operational Definitions

The following terms are used throughout this study and are defined as follows:

General Calculus course: A college-level course of instruction for the purpose of introducing the student to calculus and linear algebra concepts, particularly useful to the study of economics and business administration with special emphasis on working problems. For the purposes of this study a general calculus course is any regularly taught university-level course in this subject. College Algebra is the only prerequisite to enrollment.

Integration: Integration is the process of finding the general antiderivative of a function. Given a function f defined on some interval, if F is another function such that $F'(x)=f(x)$ for all x in this interval, we say that F is an antiderivative of f. $F(x)+c$, where c is an arbitrary constant, is the general antiderivative of f (Schelin and Bange, 1988). For the purposes of this study, integration for the control group will be the process of calculating an antiderivative using the *"properties of integration."* Integration for the experimental group will be the process of calculating an antiderivative using the *"properties of integration"* and the module, Integration: Getting it all together (Schoenfeld, 1977).

Long-term retention: According to English and English (1958), **retention** is the ability of the organism to perform a certain learned act after an interval in which the performance has not taken place. According to Gay (1992), **long-term** is any time between 2 weeks and 2 months. For the purposes of this study long-term retention was the difference in the scores between Posttest I and Posttest II.

Problem-solving strategy: English and English (1958) stated that a **problem-solving strategy** is a conscious or unconscious scheme or method for determining the answer to a problem selected from a number of alternatives. For the purposes of this study, problem-solving strategies are the specific techniques listed in the module, Integration: Getting it all together (Schoenfeld, 1977).

Test scores: For the purposes of this study a criterion-referenced assessment devised by the researcher was used to determine students' problem-solving ability for integration problems. (See Appendices E and F for Posttest I and Posttest II, respectively.)

Summary

Problem-solving in mathematics has been identified by business, government, and education as a skill that has been found to be deficient among many students. This study chose to concentrate on one specific problem-solving strategy, the choice of a particular method for solving integration problems. The question was whether presentation of this strategy would have an effect on test scores of students in a general calculus course.

REVIEW OF RELATED LITERATURE

The literature on the topic of problem-solving is generally divided into one of two main areas of research: the measurement of metacognitive behaviors during problem-solving sessions and/or the general and specific instructional strategies used to teach problem-solving. The research questions generated for this study address only the second focus of the research inquiries into this topic.

Instruction of problem-solving strategies

Polya (1945) was one of the first to concentrate on problem-solving. His book, How to solve it: A new aspect of mathematical method,, developed a strategy for general problem-solving consisting of 4 steps:

1. Understand the problem.
2. Make a plan based on how various items are connected.
3. Carry out the plan.
4. Look back at the completed solution. (Polya, 1945, p.5).

Polya's (1945) strategies are applicable to almost all disciplines, but are particularly useful in mathematics. Researchers who have utilized his approach to problem-solving strategies have produced varied results.

Studies by Smith (1988) and Zitarelli (1989) developed general problem-solving courses with conflicting results. Smith's (1988) 225 subjects were 8th-grade students who showed improvement on the Applications section of the Stanford Diagnostic Mathematics Test after training in problem-solving. Zitarelli's (1989) group consisted of gifted 4th and 5th-grade students who also received training in problem-solving strategies. However his findings, when the subjects were tested on the Applications section of the Stanford Diagnostic Mathematics Test, showed that students did improve, but not significantly.

A study by Charles and Lester (1984) involving 5th and 7th-grade students used general problem-solving strategies developed from Polya (1945). Results of their study concluded that students who received the process-oriented instructional program scored significantly higher than the control group on measures of ability to understand the problem, plan solution strategies, and get the correct results.

Starmack (1991) also used Polya's (1945) work to develop problem-solving strategies for a specific concept. Working with gifted students at a community college, he found that the effect of 21 hours of formal instruction in the techniques of proof improved students' abilities to solve complex problems.

Jenkins (1988), using 338 middle school students, compared their experience with problems solved using the strategies and direct instruction of strategies versus experience with problems solved using strategies and no direct instruction. He concluded that neither strategy improved problem-solving performance on the IPSP Problem Solving Test more than problem-solving experiences alone.

Another important cornerstone in the field of problem-solving are Schoenfeld's studies (1977, 1978, 1985) which investigated metacognitive behaviors and instructional strategies in problem-solving. His book, Mathematical problem solving (1985), is a compilation of many of his previous findings and has been the catalyst for multiple studies he and others have conducted in the use of problem-solving strategies.

Schoenfeld (1978), using a particular strategy for indefinite integration, was the catalyst for this study. The subjects were 26 students in a second quarter calculus class. Experimental and control groups were randomly chosen with 11 students receiving treatment (two students were absent when materials were handed out). The materials consisted of a workbook text, Integration: Getting it all together (Schoenfeld, 1977), and a solutions manual. No formal instruction on the strategy was provided. Results showed that the experimental group outscored the control group on six of seven questions, and by more than 10% on five of them, with the expectation of better results if the strategy was presented by the instructor (Schoenfeld, 1978).

A natural extension of the use of instructional strategies is the comparison of male and female response to the use of such aids. The research conducted into this question shows equally conflicting evidence that males and females require different techniques to satisfactorily process mathematical information.

Rosser (1989) examined test-taking differences between the sexes on the Scholastic Aptitude Test (SAT) in the area of problem-solving styles. Her sample consisted of 10 white and 10 black high school juniors from two urban and two suburban high schools in the Seattle area. The subjects were interviewed concerning their solution processes, with the conclusion that there were no major sex differences in problem-solving styles. In other words, males and females both use similar techniques for problem-solving. However, 20 subjects would not be considered an adequate sample by most research experts (Gay, 1992).

A similar study by Caporrimo (1990) investigated possible explanations for gender-related differences in mathematical abilities, *i.e.*, the fact that research repeatedly shows males more likely to excel in mathematics than females. Analyses examined the relationship of standardized mathematics achievement scores, problem-solving strategies, self-report scores, and scores on Confidence in Learning Mathematics, a survey. Subjects were 122 eighth-grade students, 70 females and 52 males. The analyses showed no gender differences in any of the scores. A comparison using the Confidence in

Learning Mathematics scores and average scores on the problem-solving strategies measure showed males exhibited a direct relationship between problem-solving scores and confidence scores, while females showed an inverse relationship.

A study in high school geometry supports the above finding. Battista (1990) examined gender differences and the role of spatial visualization in problem-solving. Findings showed that, although males and females were found to differ in spatial visualization and performance, gender was not significant in logical reasoning ability or in the use of geometric problem-solving strategies.

Engelhard (1990) looked at the relationship between gender and performance on mathematical items varying in cognitive complexity and content. The sample consisted of 1,789 female and 1,951 male Thai students and 2,040 female and 1,884 male American students. Results showed that gender affected performance in both areas, though no conclusions were drawn regarding cultural variables.

A study conducted by Weiner and Robinson (1983) attempted to determine whether cognitive abilities and personality factors were accurate predictors of mathematical achievement. The subjects were 139 gifted students, 77 males and 62 females. The only significant difference found indicated that males have a higher mathematical reasoning ability than females, with this ability being the single-best predictor of mathematical achievement for males. Verbal achievement was shown to be the best correlational predictor of mathematical achievement for females.

Summary

The literature is divided. There is no clear-cut evidence that instruction in general and/or specific problem-solving strategies improves student performance in problem-solving. Conflicting evidence regarding gender and its effect on performance and ability is also abundant.

This study investigates instructor presentation of a specific problem-solving strategy and whether it impacts upon student performance in integration tasks. Gender differences are also investigated. In addition, the effect of problem-solving strategy instruction upon long-term retention for integration tasks is examined, a topic that has not been investigated by previous researchers.

METHODOLOGY

The research questions generated for this study were:

1. *Following presentation of a problem-solving strategy to solve specific types of integration problems, will mean test scores differ significantly between experimental and control groups in a course of general calculus?*
2. *Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will gender prove significant when mean test scores for students given the experimental treatment are compared with those in the control group?*
3. *Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will there be a significant difference in long-term retention mean test scores for the control group when compared to the experimental group?*

These questions were selected for study due to differences found by previous researchers in the effectiveness of problem-solving strategies, and gender-based differences. A dearth of research was found on the effects of these problem-solving strategies and the long-term retention of material.

Subjects

The subjects utilized in this study consisted of 110 students enrolled in six sections of a General Calculus course taught during the Fall, 1992, semester at Kansas State University at Manhattan, Kansas. Due to student attrition the final number of subjects who completed all instruction and testing was 95.

Based on a biographical data sheet collected during the first week of the semester it was determined that the typical subject was 18-21 years of age, a major in Business or a Business-related area, had reached at least the Algebra II level in high school, and had completed a College Algebra course at the university level. The age range of subjects was 18-30 years old. Of the 95 subjects, 52 students received the experimental treatment (26 males and 26 females), and 43 received the control treatment (21 males and 22 females). The majority of the students were residents of Kansas.

An informed consent form was not distributed. It was determined by the Human Subjects Committee of Kansas State University that this study was exempt under Section 46.101(*b*)(*ii*) of the U.S. Code of Federal Regulations 45CFR46. All subjects were verbally informed that participation in the study was not mandatory and would in no way affect their grades.

Instruments

The instruments used for measurement were developed by the researcher. These consisted of a pretest and 2 posttests. The pretest (see Appendix C) and posttest I (see Appendix D) are identical. The posttest II (see Appendix E) is an equivalent form of the first two exams.

All measures include items to assess the student's ability to select the simplest technique for integrating a problem and implement the technique in the calculation of the integral. The instruments are criterion-referenced, being drawn from similar problems demonstrated within the classwork and assignments of students enrolled in a General Calculus course of study at Kansas State University.

Internal consistency measures of reliability were computed prior to using either of the posttest assessment instruments. Using the Kuder-Richardson formula (r), split-half internal consistency, one measure of reliability was computed using a randomly drawn sample of 25 pretests. Comparing odd answers to the even answers a reliability coefficient of .96 was determined after application of the Spearman-Brown prophecy formula (Gay, 1992), which was felt to be of adequate integrity for the purposes of this study.

Content and construct validity were established by gaining consensus of senior faculty members drawn from the Mathematics Department at Kansas State University based on their experiences in this field.

Research Design

The implied hypotheses generated for this study suggest the use of a quasi-experimental design, the nonequivalent control-group design (Campbell and Stanley, 1963). This design utilizes a pretest-multiple posttest format and will allow for the use of an arbitrary number of self-selected groups (classes). Sources of potential error in this particular design include possible regression toward the mean, though this aspect of *"learning"* is being assessed within the scope of this study by using a second posttest as an indication of exactly that aspect of learning. Gay (1992) states that other possible problems associated with this model include the interaction of subjects between selection, which is not considered to be of major importance within this study since the nature of education is often interactive, and variables such as the maturation of the students, who are generally of the same age, the history of the subjects, which is largely unknown and cannot be controlled for outside of the general characteristics common to all participants noted above, and the effects of testing, which have been incorporated within the parameters of this study as a variable of interest.

Treatment

Six different General Calculus classes scheduled during the Fall semester, 1992, were selected to participate in this study. Six different instructors with similar teaching experience were used, none of whom had previously taught this particular course. From the four morning classes, two classes with their assigned instructors were randomly selected to receive the experimental treatment with the remaining two classes and their instructors selected to receive the control treatment of standard instruction. From the two afternoon classes, one class was randomly chosen to receive the experimental treatment, while the other afternoon class received the control treatment of standard instruction. While it was not known from any previous studies whether the variable of *"time of day for instruction"* might account for some of the difference within between the groups, it was decided to incorporate it within the treatment assignment process.

The control group followed a traditional approach to the teaching of integration, based on the introduction of basic rules, with drill-and-practice used as the basis for choice of technique. Instructors used standardized lesson plans prepared by the researcher (see Appendix B) with problem examples drawn from the classroom textbook, Mathematical analysis for business and economics, 2nd ed. (Schelin and Bange, 1988), that was utilized with all groups, both experimental and control.

The experimental group used a specific strategy approach based on the module, Integration: Getting it all together (Schoenfeld, 1977), together with the classroom textbook, Mathematical analysis for business and economics, 2nd ed. (Schelin and Bange, 1988). The module was specifically adapted by the researcher using the original module, removing topics not covered in the general calculus course, and replacing problems with those more appropriate for this particular course (see Appendix G). Instructors used standardized lesson plans prepared by the researcher (see Appendix C) with the same examples from the text as those used in the control group.

Data Collection Techniques

A pretest was administered by the instructors at about the same time prior to the introduction of the topic of integration in the experimental and control group General

Calculus classes. Each instructor administered the pretest in a like and similar fashion. The pretest was done to eliminate from the study any subjects having a high level of pre-existing knowledge of the topic, and to compensate for presumed differences in the history of the subjects, a possible source of bias according to Gay (1992). Generally, a student is considered to be functioning at an independent educational level that is capable of internalizing correctly a task, when he/she has an accuracy level of 80% or higher (Walker and Shea, 1991). No subjects scored at or above an 80% level as determined by the researcher.

Posttest I was given at the conclusion of the sections on integration at about the same time for both the experimental and control group classes. The instructors administered the posttest in a similar fashion, with results scored by the researcher.

Posttest II was administered four weeks later by the instructors of the control and experimental classes and was given to assess long-term retention of the subject matter. According to Gay (1992), long-term effects are best assessed at any time between two weeks and two months. Prior to two weeks, the effects of the pretest may influence scores. After two months, the effects of maturation may be an influence. One month (four weeks) was selected as an appropriate time span to minimize both effects.

Scoring on the tests was consistent and was performed as follows for each problem:

Part I

2 points for choosing the simplest technique
3 points for choosing a correct *"u substitution"* where required
1 point for the correct *"dv"* if using integration by parts

Part II

1 point for a correct start on the problem
2 points for correct completion

Thus the total number of points per problem was as follows:

#1	2 + 3 + 1 + 2	= 8
#2	2 + 3 + 1 + 2	= 8
#3	2 + 3 + 1 + 1 + 2	= 9
#4	2 + 3 + 1 + 2	= 8
#5	2 + 1 + 2	= 5

A range of 0 to 9 points was possible depending on the problem, with a possible maximum of 38 points per test.

Analysis of Data

The technique of multiple analysis of variance (MANOVA) was used to test each of the implied hypotheses suggested by the research questions. The statistical manipulation of the data was accomplished through the use of SAS (SAS, 1985), a statistical analysis program for mainframe computers.

According to Gay (1992), MANOVA is a useful statistical test of significance when analyzing the differences between groups. An adjusted mean was used to compensate for unequal group size. Interactions between all variables were assessed to allow greater confidence in generalization of findings. As the analysis progressed, the variable *"instructor"* was found to be confounding, forcing the inclusion of a separate analysis of this factor. A .05 alpha level (α) was set for rejecting each of the implied null hypotheses.

Summary

Due to a lack of information from previous studies, it was decided to test three questions:

1. Following presentation of a problem-solving strategy to solve specific types of integration problems, will mean test scores differ significantly between experimental and control groups in a course of general calculus?
2. Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will gender prove significant when mean test scores for students given the experimental treatment are compared with those in the control group?
3. Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will there be a significant difference in long-term retention mean test scores for the control group when compared to the experimental group?

The subjects were students in six sections of a General Calculus course at Kansas State University, with three sections receiving traditional instruction (control), and three sections receiving the experimental treatment of problem-solving instruction with traditional instruction. Instruments used to assess differences in the dependent variables consisted of a pretest and two posttests, all designed by the researcher. The nonequivalent control-group design (Campbell and Stanley, 1963) was used in this study. The statistical test of significance used was analysis of variance as computed by SAS (SAS, 1985).

RESULTS

Introduction

The purpose of this study was to investigate the effects of instruction in a problem-solving strategy for integration problems. Using university-level students enrolled in different sections of a General Calculus course, results of the data concerning overall effectiveness, effectiveness by gender, and retention over time, the three research questions, are included in this chapter. A .05 alpha (α) level was used for rejecting each of the implied null hypotheses.

Results

The first research question was:

Following presentation of a problem-solving strategy to solve specific types of integration problems, will mean test scores differ significantly between experimental and control groups in a course of general calculus?

This question sought to compare test scores from Posttest I for the experimental and control groups, with an implied null hypothesis stating that there would be no statistically significant difference between mean test scores. The adjusted means were 9.4198 for the control group (treatment 1), and 14.3676 for the experimental group (treatment 2). (See Appendix G.) The standard deviation was 6.2907, with treatment proving significant at the .0003 alpha (α) level (see Appendix G). Thus the null hypothesis is rejected, and, using the adjusted mean for each group, the conclusion may be drawn that the presentation of the problem-solving strategy for an integration task increased the mean test scores significantly for the experimental group.

The second research question was:

Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will gender prove significant when mean test scores for students given the experimental treatment are compared with those in the control group?

The null hypothesis states that mean test scores for males and females will not differ based on treatment. The means for male test scores were 8.4762 for the control group and 11.2308 for the experimental group with a standard deviation of 4.8334 for the control group and a standard deviation of 6.5624 for the experimental group (see Appendix G). The means for females were 10.8636 for the control group and 17.8846 for the experimental group with standard deviations of 6.0498 and 6.8896 respectively (see Appendix G). The analysis of variance proved gender significant at the .0004 alpha level (see Appendix G). The implied null hypothesis associated with this research question is thus rejected with the assumption that the variable of gender affected mean test scores between experimental and control groups. Based on mean scores for each gender, females accounted for more of the significant difference between groups than did males.

The third research question was:

Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will there be a significant difference in long-term retention mean test scores for the control group when compared to the experimental group?

The implied hypothesis assessed long-term retention using a second posttest administered four weeks after the first posttest. The implied null hypothesis states that there will be no statistically significant difference between the long-term retention rates for integration tasks between the experimental and control groups. A variable, difference, was defined as Posttest I minus Posttest II, with all analyses made using this variable. The adjusted means were 4.1588 for the control group and 2.1254 for the experimental group with a standard deviation of 6.3373 (see Appendix G). Although means showed that the control group scores declined more than the experimental group, the difference between the two was not enough to prove statistical significance at the .05 level. Thus the null hypothesis is not rejected, with the conclusion that presentation of the problem-solving strategy had no significant effect on long-term retention for integration tasks.

Although not specifically listed as a hypothesis, all tests run on Posttest I were also performed on Posttest II. Comparison of mean test scores for experimental versus control groups (Hypothesis 1) was significant at the .0002 alpha level (see Appendix G). The variable of gender (Hypothesis 2) appeared to be insignificant at the .05 alpha level (see Appendix G), but after *post hoc* analyses of the interactions (see Appendix G) and removal of another confounding variable, *"time,"* gender proved to be significant at the .0001 alpha level (see Appendix G). Comparing male and female means showed that females again accounted for most of the significant difference between groups (see Appendix G).

Interactions between all variables were assessed on Posttest I, Posttest II, and the difference between posttests. Posttest I showed interactions between gender and treatment significant at the .0001 alpha level, which would be expected based on previous analysis (see Appendix G). Posttest II also showed interactions between gender and treatment significant at the .0001 alpha level (see Appendix G). Posttest I versus Posttest II showed no statistical significance for any of the interactions tested.

Though not specifically presented as a research question, an analysis of the variable *"instructor"* was conducted in the partitioning of the data. The assumption made by the researcher was that there might be some statistically significant differences among the various interactions tested for within the three research questions that might be attributable to certain instructor characteristics. This variable proved to be confounding, and a separate analysis was performed. Experimental and control treatments were investigated for each of the posttests in addition to the difference between posttests. In each instance, the choice of instructor was irrelevant -- there was no statistical significance found based on treatment or test (see Appendix G).

Summary

Each of the three research questions was evaluated based on results from data analysis. The first question investigated mean test scores for experimental versus control groups. A significant difference was found with the conclusion that presentation of the strategy positively affected test scores. The second question examined differences between males and females in experimental and control groups. Females in the experimental group tended to be more positively affected by the experimental treatment than were males. The third question tested long-term retention through the calculation of differences between Posttest I and Posttest II for each group. No significant difference was found at the .05 alpha level.

All tests were also performed on Posttest II, with results nearly identical to those from Posttest I. Interactions between gender and treatment were significant; choice of instructor was not significant at the .05 alpha level.

SUMMARY AND CONCLUSIONS

Summary of the Study Findings

The following research questions were generated as the focus of this study:

1. Following presentation of a problem-solving strategy to solve specific types of integration problems, will mean test scores differ significantly between experimental and control groups in a course of general calculus?

2. Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will gender prove significant when mean test scores for students given the experimental treatment are compared with those in the control group?

3. Following presentation of a problem-solving strategy to solve specific types of integration problems in a general calculus course, will there be a significant difference in long-term retention mean test scores for the control group when compared to the experimental group?

These questions were generated due, in part, to the inconclusive and contradictory findings on the part of those few researchers who have investigated aspects of this topic. Research question 3 was generated because it seemed to be a logical progression of the previous two questions and because no other research could be found that had investigated this area.

Using 110 students enrolled in six different sections of a General Calculus course at Kansas State University, Manhattan, Kansas, the researcher randomly assigned the classes to either control group or experimental group status. The control group received traditional instruction in integration tasks. The experimental group received the same traditional instruction along with a specific problem-solving strategy approach based on Integration: Getting it all together (Schoenfeld, 1977), and specifically adapted for the general calculus class by the researcher. The instructors for the six classes received instructions and followed standardized lesson plans. Both the control and the experimental groups took a pretest to determine pre-existing knowledge levels of the subjects for the task of integration. After the presentation of the instructional unit the subjects were administered Posttest I. Four weeks later the subjects were administered Posttest II.

Analyses of the posttests of the 95 subjects who remained at the conclusion of the research using multiple analysis of variance as the statistical test of significance showed that students presented with a specific strategy for integration tended to have higher mean test scores than those not exposed to the strategy approach. Test scores were extremely low, possibly due to time constraints, and the fact that students were not informed of the impending exam. In addition, it was found that gender was significant in a comparison between the experimental and control groups. Specifically, it was found that females benefitted more from the experimental treatment than did male students. Tests also indicated that the presentation of the problem-solving strategy did not affect long-term retention at a statistically significant level.

Conclusions

Based on the results of this study, the following conclusions may be made:

1. Instruction in a problem-solving strategy for integration tasks benefitted those subjects receiving this type of instruction when it was paired with traditional instruction as opposed to those subjects who received traditional instruction only.
2. Females tended to benefit more than males from instruction in problem-solving strategies. This would tend to support the ongoing research (Schwartz and Reisberg, 1991) that has cautiously stated that females, due to an earlier onset of puberty and the associated maturational differences within the cerebral cortex, may benefit more from a language-based or left-brain approach to mathematics instruction.
3. The findings of this study showed that long-term retention of integration tasks was no better among those who received problem-solving strategy instruction with traditional instruction than for those subjects who received only traditional instruction.

Limitations of the study

There are potential limitations to this study. Among the potential limitations are the use of self-selection, *i.e.*, students selected the class of General Calculus based on factors such as personal schedule, availability, etc., and while unavoidable, complete randomization did not occur and is a limiting factor in terms of the generalizability of the results. While there were no noticeable differences between the classes, as a compensation the classes were randomly assigned to either treatment or control status. The pretest, analysis of variance, and analysis of biographical data reduced, but did not eliminate differences.

The pretest itself may be a limiting factor in that it may be a sensitizing element, again reducing generalizability. However, in an educational setting where a standard form of assessment is the paper-and-pencil test, it is expected that students, the subjects used within this investigation, will be less sensitive to the stimuli of a pretest.

Maturation, mortality, and interaction between groups may be additional threats to the validity of the study, but not exceedingly so. The subjects were adults and the effect of maturation is minimized more among this group according to Gay (1992). Mortality was a factor within this study. The original sample of 110 was reduced to 95 due to attrition and subjects who were not available for all testing sessions. The interaction between groups, which could not be controlled in a real-life situation, is also unknown and a potential threat to the validity of the study.

The selection of instructors may also be a limitation. Although analyses showed the choice of instructors to be irrelevant, other factors not immediately available to the researcher may impact the study, *i.e.*, hidden biases.

The test instrument is another possible limitation within this study. Having been developed by the researchers, the tests may contain a bias toward the desired outcome. Content and construct validity were established by expert opinion derived from the consensus of the faculty at Kansas State University only, possibly, but not probably, compromising the overall validity of the instrument.

The most important limiting factor in this study is the subjects. Any study dealing with human behaviors is likely to be suspect due to the very nature of human subjects. Factors such as intelligence, study habits, motivation, etc. were unknown factors in this study and could have contributed to the results.

Questions for future research

The results of this study support the finding that instructor presentation of a problem-solving strategy for integration tasks does increase test scores of General Calculus students. Additional research questions that evolve from this finding include the use of other problem-solving strategies in other areas of mathematics. *Do problem-solving strategies facilitate learning in geometry or matrix analyses? Do problem-solving strategies benefit complex mathematics more than simple mathematics, or is it beneficial to all forms?*

The issue of gender differences is significant in this study and definitely warrants further investigation. *Are there actual differences between males and females in the way that mathematical concepts are best learned? Are there actual differences between males and females in the conceptualization of mathematics?* Attempts could also be made to measure attitudes and metacognitive behaviors in this area. *Are problem-solving strategies more useful that other pedagogical methods of instruction?* Additional research is advised so that we as educators might become aware of the most successful methods for student instruction.

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APPENDIX A

LESSON PLANS (CONTROL GROUP)

INSTRUCTIONS FOR CONTROL GROUP INSTRUCTORS

Give Pretest Exam before starting Section 6.1 in Mathematical Analysis for Business and Economics (Schelin and Bange, 1988) - students are allowed 15 minutes, but most will know nothing about integration so it will not take them long. On Part I, ask, if they specify substitution or integration by parts, that they also indicate what substitutions they will make.

In general, all examples need to be covered.

Posttest Exam I will be handed out later - it will be administered following completion of Section 7.2 in the text. Posttest Exam II will also be handed out later - it will be administered approximately one to two weeks before the end of the semester.

All exams will be graded by the researcher. If you wish to use one of the Posttest exams as a quiz, the scoring system could be adapted for such use.

SECTION 6.1 - ANTIDERIVATIVE (CONTROL GROUP)

Objectives:

The student should be able to:
1. Give the definition of and notation for the antiderivative.
2. Calculate antiderivatives using the *"Basic Rules of Integration."*
3. Calculate antiderivatives dealing with sums and simple rational functions.
4. Solve an application problem involving integrals.

Lesson Plan

Def. 6.1 - Let f be a function defined on some interval. If F is another function such that

$$F'(x) = f(x)$$

for all x in this interval, we say that F is an antiderivative of f.

If F is an antiderivative of f, than F + c, where c is a constant, is also an antiderivative of f. It is also true that any antiderivative G of f can be written as G(x) = F(x) + c. We call F(x) + c the general antiderivative of f.

If F is a differentiable function, then the differential of F(x) + c is

$$dF = F'(x)\, dx.$$

Introduce the symbol $\int$ (integral sign), which represents the inverse of the differential, that is,

$$dF = F(x) + c.$$

Then

$$\int f(x)\, dx = F(x) + c.$$

So $\int f(x)\, dx$:notes all antiderivatives of function f (provided f has an antiderivative).

Basic Rules of Integration

1. $\int k\,dx = kx + c$, *for any constant k.*

2. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$, *for* $n \neq -1$.

3. $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + c.$

4. $\int e^x\,dx = e^x + c.$

5. $\int kf(x)\,dx = k\int f(x)\,dx$, *for constant k.*

6. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$

Examples

*Ex.*1 (*no.*5) $\int x^{\frac{2}{5}}\,dx = \frac{1}{\frac{2}{5}+1}x^{\frac{2}{5}+1} + c$

$$= \frac{5}{7}x^{\frac{7}{5}} + c$$

$$Ex.2\ (no.6)\quad \int z^{-\frac{3}{4}}dz = \frac{1}{-\frac{3}{4}+1} z^{-\frac{3}{4}+1} + c$$

$$= 4z^{\frac{1}{4}} + c$$

$$Ex.3\ (no.9)\quad \int (5x^2 - 3x + 4)\,dx = \int 5x^2 dx - \int 3x\,dx + \int 4\,dx$$

$$= \frac{5}{3}x^3 - \frac{3}{2}x^2 + 4x + c$$

$$Ex.4\ (no.11)\quad \int (x+3)(2x-1)\,dx = \int (2x^2 + 5x - 3)\,dx$$

$$= \int 2x^2 dx + \int 5x\,dx - \int 3\,dx$$

$$= \frac{2}{3}x^3 + \frac{5}{2}x^2 - 3x + c$$

$$Ex.5\ (no.22)\quad \int \frac{x+5}{x}dx = \int \left(\frac{x}{x} + \frac{5}{x}\right)dx$$

$$= \int 1\,dx + \int \frac{5}{x}dx$$

$$= x + 5\ln x + c$$

$$Ex.6\ (no.24)\quad \int \frac{x^2+3}{\sqrt{x}}dx = \int x^{\frac{3}{2}}dx + \int 3x^{-\frac{1}{2}}dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + c$$

*Ex.*7 (*no.*32) $MC(q) = 18{,}000 - 50q - .06q^2, 0 \le q \le 150$
fixed costs \$850 (*this is the constant*),
what is total cost?

$$
\begin{aligned}
C(q) = \int MC(q)\,dq &= \int (18000 - 50q - .06q^2)\,dq \\
&= \int 18000\,dq - \int (50q)\,dq - \int (.06q^2)\,dq \\
&= 18000q - 25q^2 - .02q^3 + 850 \\
C(30) &= 18000(30) - 25(30)^2 - .02(30)^3 + 850 \\
&= \$517{,}810.
\end{aligned}
$$

SECTION 6.2 - THE DEFINITE INTEGRAL (CONTROL GROUP)

Objectives:

The student should be able to:
1. Recognize a definite integral using the definition.
2. Calculate a definite integral using formula 6.6 and theorem 6.1.
3. Solve an application problem with a definite integral.

Lesson Plan:

Def. 6.2 - If f is a continuous function and F is an antiderivative of f, then the change in F, when the independent variable changes from x=a to x=b. is denoted by

$$\int_a^b f(x)\,dx$$

and called the definite integral of f from a to b. The values a and b are called limits of integration. Let $F(x)]_a^b$ denote this difference so that

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

Examples

*Ex.*1 (*no*.3) $\int_{2}^{10}(2x-1)\,dx = \int_{2}^{10}(2x)\,dx - \int_{2}^{10}(1)\,dx$

$$= 2\int_{2}^{10}(x)\,dx - \int_{2}^{10}(1)\,dx$$

$$= 2[\frac{1}{1+1}x^{1+1}]_{2}^{10} - [\frac{1}{0+1}x^{0+1}]_{2}^{10}$$

$$= 2[\frac{1}{2}x^{2}]_{2}^{10} - [x]_{2}^{10}$$

$$= 2[50-2] - [10-2]$$

$$= 96 - 8$$

$$= 88$$

*Ex.*2 (*no*.9) $\int_{0}^{3}(y^2-6y+4)\,dy = \int_{0}^{3}(y^2)\,dy - 6\int_{0}^{3}(y)\,dy + 4\int_{0}^{3}(1)\,dy$

$$= [\frac{1}{2+1}y^{2+1}]_{0}^{3} - 6[\frac{1}{1+1}y^{1+1}]_{0}^{3} + 4[y]_{0}^{3}$$

$$= [\frac{1}{3}y^{3}]_{0}^{3} - 6[\frac{1}{2}y^{2}]_{0}^{3} + 4[y]_{0}^{3}$$

$$= 9 - 27 + 12$$

$$= -6$$

Theorem 6.1 Properties of the Definite Integral:

1. $\int_a^a f(x)\,dx = 0$

2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *, for any c with a < c < b.*

Examples

*Ex.*3 (*no.*15) $\int_1^5 \frac{1}{x^2}\,dx = \int_1^5 x^{-2}\,dx = [\frac{1}{-2+1}x^{-2+1}]_1^5$

$$= [-x^{-1}]_1^5$$

$$= [-\frac{1}{x}]_1^5$$

$$= [-\frac{1}{5}-(-1)]$$

$$= \frac{4}{5}$$

*Ex.*4 (*no*.26) $\int_{-2}^{-1} \frac{5x^2+x+3}{x^2}\,dx = \int_{-2}^{-1} (5+\frac{1}{x}+\frac{3}{x^2})\,dx$

$= 5\int_{-2}^{-1} dx + \int_{-2}^{-1} x^{-1}\,dx + 3\int_{-2}^{-1} x^{-2}\,dx$

$= 5\,[x]_{-2}^{-1} + [\ln x\,]_{-2}^{-1} + 3\,[-x^{-1}]_{-2}^{-1}$

$= 5\,(-1+2) + (\ln 1 - \ln 2) + 3\,(1-\frac{1}{2})$

$= 5 + (0 - .693) + 1.5$

$= 5.807$

*Ex.*5 (*no*.30) $\int_{1}^{25} \frac{x^2+3x-1}{\sqrt{x}}\,dx = \int_{1}^{25} (x^2+3x-1)(x^{-\frac{1}{2}})\,dx$

$= \int_{1}^{25} (x^{\frac{3}{2}} + 3x^{\frac{1}{2}} - x^{-\frac{1}{2}})\,dx$

$= \int_{1}^{25} x^{\frac{3}{2}}\,dx + 3\int_{1}^{25} x^{\frac{1}{2}}\,dx - \int_{1}^{25} x^{-\frac{1}{2}}\,dx$

$= \frac{2}{5}x^{\frac{5}{2}}\Big]_{1}^{25} + 3\,[\frac{2}{3}x^{\frac{3}{2}}]_{1}^{25} - [2x^{\frac{1}{2}}]_{1}^{25}$

$= 1250 - \frac{2}{5} + 250 - 2 - 10 + 2$

$= 1489.6$

*Ex.*6 (*no*.33) $MC = 300 - 1.4q - 6\sqrt{q},\ 50 \le q \le 120$

$\int_{90}^{100} 300 - 1.4q - 6\sqrt{q}\,dq = [300q - .7q^2 - 4q^{\frac{3}{2}}]_{90}^{100}$

$= 30000 - 7000 - 4000 - 27000 + 5670 + 3415.26$

$= \$\,1085.26$

SECTION 6.3 AREA (CONTROL GROUP)

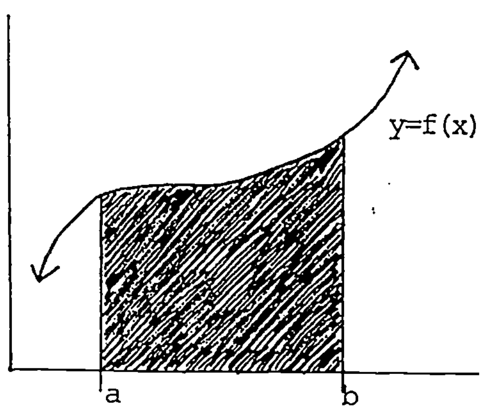
Objectives:

The student should be able to:
1. Find the area of a bounded, single function region using the definite integral.
2. Find the area of a region bounded by two functions using the definite integral.

Lesson Plan

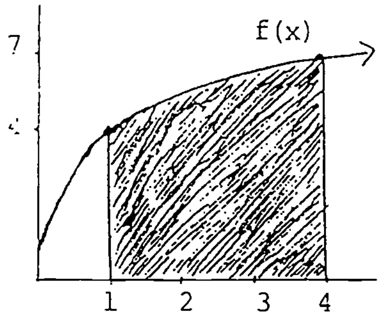
Theorem 6.2 - If f is positive and continuous on [a,b], then the area of the region bounded by y = f(x), x=a, x=b, and the x-axis is given by the definite integral

$$\int_a^b f(x)\,dx$$



Examples

*Ex.*1 (*no.*3) $f(x) = 1+3\sqrt{x};\ a=1,\ b=4$



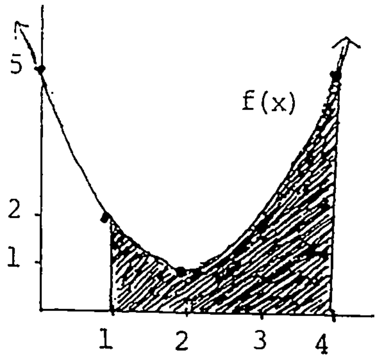
$$\int_1^4 (1+3\sqrt{x})\,dx = [x+2x^{\frac{3}{2}}]_1^4$$

$$= [4+2(8)] - [1+2(1)]$$

$$= 20 - 3$$

$$= 17$$

*Ex.*2 (*no.*5) $f(x) = x^2 - 4x + 5;\ a=1, b=4$



$$-\frac{b}{2a} = \frac{-(-4)}{2(1)} = 2$$

$$f(2) = 4-8+5 = 1$$

$$vertex = (2,1)$$

$$\int_1^4 (x^2-4x+5)\,dx = [\frac{1}{3}x^3 - 2x^2 + 5x]_1^4$$

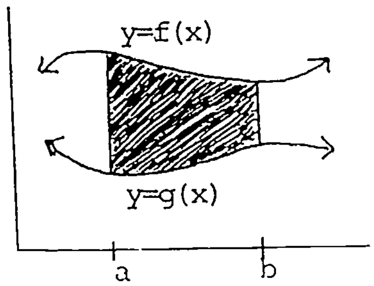
$$= [(\frac{1}{3})(4)^3 - (2)(4)^2 + 5(4)] - [(\frac{1}{3})(1)^3 - 2(1)^2 + 5(1)]$$

$$= \frac{64}{3} - 32 + 20 - \frac{1}{3} + 2 - 5$$

$$= 6$$

<u>Theorem 6.3</u> - If f and g are continuous functions on [a,b], and if $f(x) \geq g(x)$ for all x in [a,b], then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, and between $x=a$ and $x=b$ is given by the definite integral

$$\int_a^b [f(x) - g(x)]\,dx.$$



Examples

Ex.3 (*no*.11) $f(x) = x^2 - x + 5; g(x) = \sqrt{x}; a = 1, b = 4$

$f(1) = 5,\ \ g(1) = 1,\ \ so\ f(1) \geq g(1)$

$f(4) = 17,\ \ g(4) = 2,\ \ so\ f(4) \geq g(4)$

$$\int_1^4 [(x^2 - x + 5) - (\sqrt{x})]\, dx = [\frac{1}{3}x^3 - \frac{1}{2}x^2 + 5x - \frac{2}{3}x^{\frac{3}{2}}]_1^4$$

$$= \frac{64}{3} - 8 + 20 - \frac{16}{3} - \frac{1}{3} + \frac{1}{2} - 5 + \frac{2}{3}$$

$$= \frac{143}{6}$$

Ex.4 (*no*.21) $f(x) = \frac{1}{x}, g(x) = \frac{5}{4} - \frac{1}{4}x$

Set f(x) = g(x) for intersection points

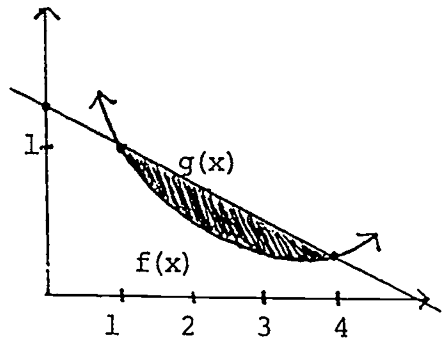
$$\frac{1}{x} = \frac{5}{4} - \frac{x}{4}$$

$$4 = 5x - x^2$$

$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1)\ 0$$

$$x = 4\ \ x = 1$$



$$\int_1^4 [\frac{5}{4} - \frac{1}{4}x - \frac{1}{x}]\, dx = [\frac{5}{4}x - \frac{1}{8}x^2 - \ln x]_1^4$$

$$= [(\frac{5}{4})(4) - (\frac{1}{8})(4)^2 - \ln 4] - [(\frac{5}{4})(1) - (\frac{1}{8})(1)^2 - \ln 1]$$

$$= 5 - 2 - \ln 4 - \frac{5}{4} + \frac{1}{8} + 0$$

$$= \frac{15}{8} - \ln 4$$

SECTION 6.4 - THE METHOD OF SUBSTITUTION (CONTROL GROUP)

Objectives:

The student should be able to:

1. Calculate an indefinite integral using the method of substitution.
2. Calculate a definite integral using the method of substitution.

Lesson Plan

Definition - Making a substitution in the integrand to aid in finding an antiderivative is called the method of substitution.

Examples

$Ex.1\ (no.3)\ \int (2x+5)^3 dx$

$let u = 2x+5,\ du = 2\,dx$

$$\int (2x+5)^3 dx = \int (2x+5)^3 \left(\frac{1}{2}\right)(2) dx$$
$$= \frac{1}{2}\int u^3 du$$
$$= \frac{1}{2}\left(\frac{1}{4}u^4\right) + c$$
$$= \frac{1}{8}(2x+5)^4 + c$$

*Ex.*2 (*no*.8) $\int s^5(4-3s^6)\,ds$

$$let\ u = 4-3s^6,\ du = -18s^5 ds$$

$$\int s^5(4-3s^6)\,ds = \int (-\frac{1}{18})(-18)s^5(4-3s^6)\,ds$$

$$= -\frac{1}{18}\int u\,du$$

$$= -\frac{1}{18}(\frac{1}{2}u^2) + C$$

$$= -\frac{1}{36}(4-3s^6)^2 + C$$

*Ex.*3 (*no*.27) $\int \frac{6r\,dr}{r^2+4}$

$$let\ u = r^2+4,\ du = 2r\,dr$$

$$\int \frac{6r\,dr}{r^2+4} = \int (\frac{1}{r^2+4})(6r)\,dr$$

$$= \int (\frac{1}{r^2+4})(3)(\frac{1}{3})(6r)\,dr$$

$$= 3\int \frac{1}{u}\,du$$

$$= 3\ln\ u\ \ + C$$

$$= 3\ln\ r^2+4\ \ + C$$

*Ex.*4 (*no*.36) $\int \frac{\sqrt{\ln x}}{x}\,dx$

$$let\ u = \ln x,\ du = \frac{1}{x}dx$$

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \frac{1}{x}\sqrt{\ln x}\,dx$$

$$= \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}(\ln x)^{\frac{3}{2}} + C$$

$Ex.5$ $(no.37)$ $\int \frac{\ln^4(3-x)}{3-x}dx$

$$let\ u = \ln(3-x),\ du = \frac{1}{3-x}dx$$

$$\int \frac{\ln^4(3-x)}{3-x}dx = \int u^4 du$$

$$= \frac{1}{5}u^5 + C$$

$$= \frac{1}{5}\ln^5(3-x) + C$$

$Ex.6$ $(no.40)$ $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

$$let\ u = e^x + e^{-x},\ du = e^x - e^{-x}dx$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int \frac{du}{u}$$

$$= \int \frac{1}{u}du$$

$$= \ln u \ + C$$

$$= \ln e^x + e^{-x} \ + C$$

$Ex.7$ $(no.49)$ $\int_0^2 \frac{dx}{8x+1}$

$$let\ u = 8x+1,\ du = 8\,dx$$

$$\int_0^2 \frac{dx}{8x+1} = \int_0^2 (\frac{1}{8x+1})(\frac{1}{8})(8)\,dx$$

$$= \frac{1}{8}\int \frac{1}{u}du$$

$$= \frac{1}{8}[\ln u \ ]$$

$$= \frac{1}{8}[\ln 8x+1 \ ]_0^2$$

$$= \frac{1}{8}[\ln 17 - \ln 1]$$

$$= \frac{1}{8}\ln 17$$

SECTION 6.5 - APPLICATIONS OF THE DEFINITE INTEGRAL
(CONTROL GROUP)

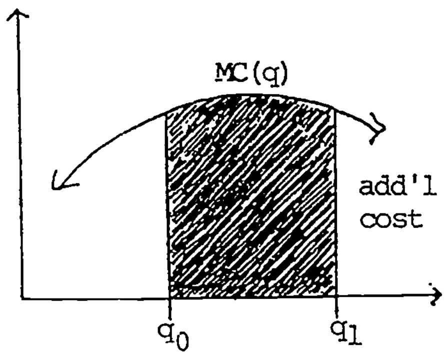
Objectives:

The student should be able to:

1. Find increased cost given marginal cost.
2. Find increased revenue given marginal revenue.
3. Find the equilibrium point, producer's surplus, and consumer's surplus given supply and demand functions.
4. Find the mean value of a function over a given interval.

Lesson Plan

Increased Cost - area of a region bounded by marginal cost function.



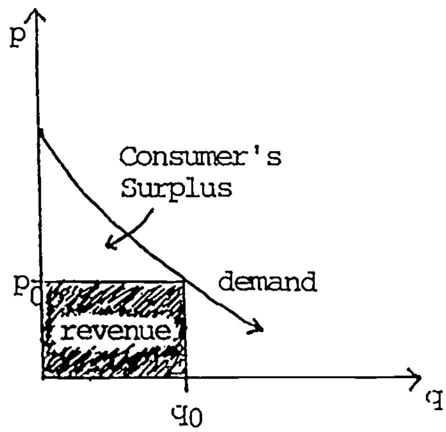
(Figure 6.9)

Example

*Ex.*1 (*no*.4) $MC = 10 - .08q + .006q^2, 0 \le q \le 150$
increase q, 100 – 120, find increased cost.

$$\int_{100}^{120} MC(q) = \int_{100}^{120} (10 - .08q + .006q^2)\,dq$$
$$= [10q - .04q^2 + .002q^3]_{100}^{120}$$
$$= 4080 - 2600$$
$$= \$1480$$

Consumer's Surplus - area of region below demand function but above horizontal line $p = p_0$, representing money not spent by consumers who would have been willing to pay a price higher than p_0 for the product.



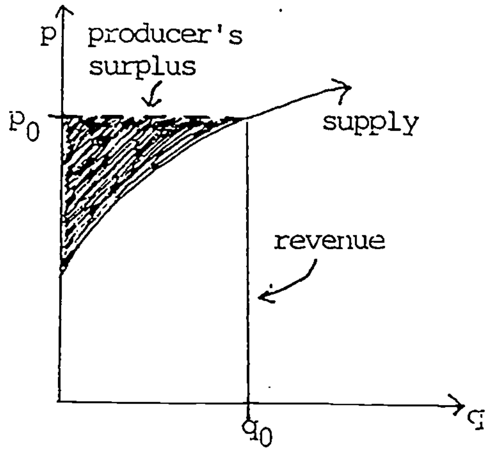
(Figure 6.10)

Producer's Surplus - area of region above supply curve and below horizontal line $p = p_0$,

$$CS = \int_0^{q_0} [f(q) - p_0]\, dq$$

where p_0 is selling price, q_0 is demand level,
$p = f(q)$ is demand function.

representing money that suppliers would not have received if demand had been less than q_0.



(Figure 6.11)

$$PS = \int_0^{q_0} [p_0 - f(q)]\, dq$$

where $p = f(q)$ is supply function,
q_0 is demand level.

Examples

*Ex.*2 (*no.*7) *Supply*: $p = 2.8 + .5q - .002q^2$

Demand: $p = 25 - .36q + .002q^2$

$0 \le q \le 90$

Find equilibrium point (where supply = demand), Consumer's Surplus, Producer's Surplus.

$$2.8 + .5q - .002q^2 = 25 - .36q + .002q^2$$

$$0 = .004q^2 - .86q + 22.2$$

$$q = \frac{-(-.86) \pm \sqrt{(-.86)^2 - 4(.004)(22.2)}}{(2)(.004)}$$

$$= \frac{.86 \pm \sqrt{.7396 - .3552}}{.008}$$

$$= \frac{.86 \pm .62}{.008}$$

$q = 185$, $q = 30$, *but* $0 \le q \le 90$ *so* $q = 30$.

for $q = 30$,

$$p(30) = 2.8 + .5(30) - .002(30)^2 = 16 = p$$

$$PS = \int_0^{30} [16 - (2.8 + .5q - .002q^2)]\,dq$$

$$= \int_0^{30} (13.2 - .5q + .002q^2)\,dq$$

$$= [13.2q - .25q^2 + \frac{.002}{3}q^3]_0^{30}$$

$$= (189 - 0)$$

$$= \$189$$

$$CS = \int_0^{30} [(25 - .36q + .002q^2) - 16]\,dq$$

$$= \int_0^{30} (.002q^2 - .36q + 9)\,dq$$

$$= [\frac{.002}{3}q^3 - .18q^2 + 9q]_0^{30}$$

$$= \$126$$

*Ex.*4 (*no.*27) $V = 50{,}000e^{-.3t},\ 0 \le t \le 10$

$$MV = \frac{1}{10-0}\int_0^{10} 50{,}000e^{-.3t}\,dt$$

$$= \frac{1}{10}\left[(50{,}000)(e^{-.3t})\left(\frac{1}{-.3}\right)\right]_0^{10}$$

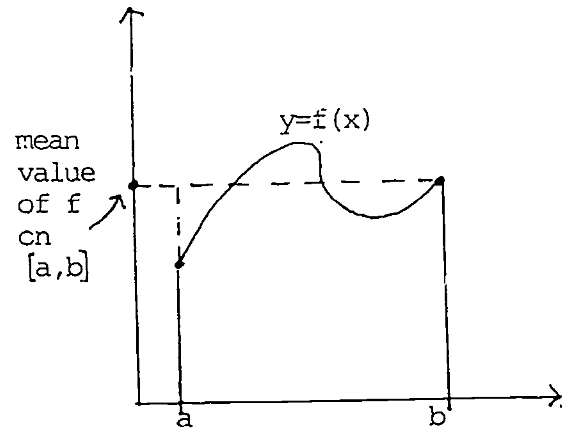
$$= \frac{1}{10}\left[-166666.7e^{-.3t}\right]_0^{10}$$

$$= \frac{1}{10}(-8297.85 + 166666.7)$$

$$= \$15836.89$$

Mean Value - the height of a rectangle whose area equals that of the region bounded above by y = f(x) between x=a and x=b.

$$MV = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Examples

*Ex.*3 (*no.*17) $g(x) = x^3$ *on* $[-1,1]$

$$MV = \frac{1}{1-(-1)}\int_{-1}^{1} x^3\,dx$$
$$= \frac{1}{2}\int_{-1}^{1} x^3\,dx$$
$$= \frac{1}{2}[\frac{1}{4}x^4]_{-1}^{1}$$
$$= \frac{1}{2}(\frac{1}{4} - \frac{1}{4})$$
$$= 0$$

SECTION 7.1 - RIEMANN SUMS AND NUMERICAL INTEGRATION (CONTROL GROUP)

Objectives:

The student should be able to:
1. Evaluate the Riemann sum for a function over some interval given values of N.
2. Estimate the value of a given definite integral using the midpoint rule.
3. Use the trapezoidal rule to estimate the value of a given integral.
4. Use the trapezoidal rule to estimate a definite integral given a set of tabular data.

Lesson Plan

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

when f is continuous on [a,b] and F is an antiderivative of f.

Riemann Sum - approximates the area under a curve by partitioning the interval into subintervals, replacing the graph of the function with an approximating horizontal segment over each subinterval, and summing the areas of the resulting rectangles. To construct N subdivisions of equal width Δx given

$$\int_a^b f(x)\,dx,\ take\ \Delta x = \frac{b-a}{N}.$$

For the height of each rectangle use f(c) for some c in the subinterval. The area of the rectangle is then $f(c) \cdot \Delta x$. If c_1 is selected in the first subinterval, c_2 in the second, and so on, the sum of these areas

$$S = f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_N)\Delta x$$

is an approximation for $\int_a^b f(x)\,dx$. *See figure* 7.2.

Examples

Ex.1 (*no*.3) $f(x) = 3x^2 + 1$ *on* [0,4]; $N = 5$,

$c_1 = 0, c_2 = 1, c_3 = 2, c_4 = 3, c_5 = 3.8.$

$$(estimate \int_0^4 (3x^2+1)\,dx).$$

$$\Delta x = \frac{4-0}{5} = .8$$

$$f(c_1) = f(0) = 1; f(c_2) = f(1) = 4$$

$$f(c_3) = f(2) = 13; f(c_4) = f(3) = 28$$

$$f(c_5) = f(3.8) = 44.3$$

$$S = (1)(.8) + (4)(.8) + (13)(.8) + (28)(.8) + (44.3)(.8)$$
$$= 72.24$$

$$\int_0^4 (3x^2+1)\,dx = [x^3 + x]_0^4$$
$$= [(4)^3 + 4] - 0$$
$$= 68$$

$$Ex.2\ (no.7)\quad f(x) = \frac{x}{x^2+1}\ on\ [1,4];\ N=3,$$

$$c_1 = 1.1,\ c_2 = 2.5,\ c_3 = 3.6$$

$$\Delta x = \frac{4-1}{3} = 1$$

$$f(c_1) = .4977;\ f(c_2) = .3448;\ f(c_3) = .2579$$

$$S = [.4977 + .3448 + .2579](1) = 1.1$$

$$\int_1^4 \frac{x}{x^2+1}\,dx$$

$$let\ u = x^2+1,\ du = 2x\,dx$$

$$\int_1^4 \frac{x}{x^2+1}\,dx = \int_1^4 \frac{1}{2}\left(\frac{2x}{x^2+1}\right)dx$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}[\ln u]$$

$$= \frac{1}{2}[\ln x^2+1]_1^4$$

$$= \frac{1}{2}[\ln 17 - \ln 2]$$

$$= \frac{1}{2}\ln\frac{17}{2}$$

$$= \frac{1}{2}\ln 8.5$$

<u>Midpoint Rule</u> - specialized Riemann sum used to approximate definite integrals. Each c_i is selected as the midpoint of the subinterval.

Example

$$Ex.3\ (no.13)\ \int_1^3 \frac{1}{x}\,dx, N=5$$

$$\Delta x = \frac{3-1}{5} = \frac{2}{5} = .4$$

subintervals are

[1,1.4], [1.4,1.8], [1.8,2.2], [2.2,2.6], [2.6,3.0]

$$c_1 = 1.2, c_2 = 1.6, c_3 = 2.0, c_4 = 2.4, c_5 = 2.8$$

$$S = [\frac{1}{1.2} + \frac{1}{1.6} + \frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.8}](.4)$$

$$= (2.7321)(.4)$$

$$= 1.0929$$

Trapezoidal Rule - approximates a definite integral using the sum of areas of trapezoids rather than rectangles. (See figure 7.5.)

Use right endpoint values to form

$$S_R = [f(c_1) + f(c_2) + \ldots + f(c_{N-1}) + f(b)]\Delta x$$

Use left endpoint values to form

$$S_L = [f(a) + f(c_1) + f(c_2) + \ldots + f(c_{N-1})]\Delta x.$$

Then the Trapezoidal Rule is:

$$T = \frac{S_L + S_R}{2}$$

$$= \frac{\Delta x}{2}[f(a) + 2(f(c_1) + f(c_2) + \ldots + f(c_{N-1})) + f(b)]$$

Examples

*Ex.*4 (*no*.19) $\int_0^2 \sqrt{1+x^3}\,dx$ *for* $N=5$

$$\Delta x = \frac{2-0}{5} = \frac{2}{5} = .4$$

subintervalsare

[0,.4], [.4,.8], [.8,1.2], [1.2,1.6], [1.6,2.0]

$$t_5 = .2(1+2(1.03+1.23+1.65+2.26)+3)$$
$$= 3.268$$

$$\textit{for } N = 10,\ \Delta x = \frac{2-0}{10} = \frac{1}{5} = .2$$

subintervals are

[0,.2], [.2,.4], [.4,.6], [.6,.8], [.8,1.0],
[1.0,1.2], [1.2,1.4], [1.4,1.6], [1.6,1.8], [1.8,2.0]

$$T_{10} = .1(1+2(1+1.03+1.1+1.23+1.4+1.7+1.9+2.3+2.6)+3)$$
$$= .1(1+2(14.26)+3)$$
$$= 3.252$$

Ex.5 (*no*.21) $R(100) - R(50) = \int_{50}^{100} MR(q)\,dq$

$$\Delta q = 10, N = 5$$

$$\int_{50}^{100} MR(q)\,dq = \frac{10}{2}[MR(50) + 2(MR(60) + MR(70) + MR(80) + MR(90)) + MR(100)]$$
$$= 5[20 + 2(15 + 12 + 8 + 4) + 2]$$
$$= 5(20 + 78 + 2)$$
$$= 500$$

SECTION 7.2 - INTEGRATION TECHNIQUES (CONTROL GROUP)

Objectives:

The student will be able to:
1. Find an antiderivative using integration by parts.
2. Use Table 7.1 to find antiderivative.

Lesson Plan

Integration by parts - based on the product rule for derivatives.

$$\int u\,dv = uv - \int v\,du$$

Use the above formula if it is possible to find a function v whose differential is dv and if ∫v du is simpler than the original antiderivative.

Examples

$$Ex.1\ (no.8)\quad \int \sqrt{x}\ln x\,dx$$

$$let\, u = \ln x,\ dv = \sqrt{x}\,dx$$

$$then\ du = \frac{1}{x}dx,\ v = \int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}$$

$$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}x^{-1}dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}(\frac{2}{3})x^{\frac{3}{2}} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$$

*Ex.*2 (*no*.9) $\int x(2x+1)^4 dx$

$$\text{let } u = x,\ dv = (2x+1)^4 dx$$

$$\text{then } du = dx,\ v = \int (2x+1)^4 dx = \frac{(2x+1)^5}{5}\left(\frac{1}{2}\right)$$

$$\int x(2x+1)^4 dx = x\left[\left(\frac{1}{2}\right)\frac{(2x+1)^5}{5}\right] - \int \left(\frac{1}{10}(2x+1)^5\right) dx$$

$$= \frac{1}{10}x(2x+1)^5 - \frac{1}{10}\left[\frac{(2x+1)^6}{6}\left(\frac{1}{2}\right)\right] + C$$

$$= \frac{1}{10}x(2x+1)^5 - \frac{1}{120}(2x+1)^6 + C$$

Introduce List of Antiderivatives - Table 7.1, p. 271.
It is usually necessary to make a substitution or a change of variable to put the given integrand into one of the forms found in the table.

Examples

*Ex.*3 (*no*.17) $\int x^2 4^{6-x^3} dx$

$$\text{use no. } 4,\ \int b^u du = \frac{1}{\ln b} b^u + C$$

$$\text{let } u = 6 - x^3,\ du = -3x^2 dx$$

$$\int x^2 4^{6-x^3} dx = \int 4^{6-x^3}\left(-\frac{1}{3}\right)(-3x^2 dx)$$

$$= -\frac{1}{3}\int 4^u du$$

$$= \left(-\frac{1}{3}\right)\left[\frac{1}{\ln 4}(4^{6-x^3})\right] + C$$

Ex.4 (no.18) $\int \frac{dx}{25x^2-9} = \int \frac{dx}{(5x)^2-(3)^2}$

$$\text{use no. 14, } \int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\frac{u-a}{u+a} + C$$

$$\text{let } u = 5x,\ du = 5\,dx,\ a = 3$$

$$\int \frac{dx}{25x^2-9} = \int \left(\frac{1}{5}\right)\left(\frac{5\,dx}{(5x)^2-(3)^2}\right)$$

$$= \frac{1}{5}\left[\frac{1}{2(3)}\ln\frac{5x-3}{5x+3}\right] + C$$

$$= \frac{1}{30}\ln\frac{5x-3}{5x+3} + C$$

Ex.5 (no.27) $\int_{-1}^{2} x e^{\frac{x}{2}}\,dx$

$$\text{use no. 5, } \int u e^{u}\,du = ue^{u} - e^{u} + C$$

$$\text{let } u = \frac{1}{2}x,\ du = \frac{1}{2}dx$$

$$\int_{-1}^{2} x e^{\frac{x}{2}}\,dx = \int_{-1}^{2} (2)\left(\frac{1}{2}x\right)e^{\frac{x}{2}}(2)\left(\frac{1}{2}dx\right)$$

$$= \int 4ue^{u}\,du$$

$$= 4\int ue^{u}\,du$$

$$= 4[ue^{u} - e^{u}]$$

$$= 4\left[\frac{x}{2}e^{\frac{x}{2}} - e^{\frac{x}{2}}\right]_{-1}^{2}$$

$$= 4\left[\left(\frac{2}{2}e^{\frac{2}{2}} - e^{\frac{2}{2}}\right) - \left(-\frac{1}{2}e^{-\frac{1}{2}} - e^{-\frac{1}{2}}\right)\right]$$

$$= 4\left(\frac{3}{2}e^{-\frac{1}{2}}\right)$$

$$= 6e^{-\frac{1}{2}}$$

APPENDIX B

LESSON PLANS (EXPERIMENTAL GROUP)

INSTRUCTIONS FOR EXPERIMENTAL GROUP INSTRUCTORS

Give Pretest Exam before starting Section 6.1 in Mathematical Analysis for Business and Economics (Schelin and Bange, 1988) - students are allowed 15 minutes but most will know nothing about integration so it will not take them long. On Part I, ask that they indicate what substitutions they will make when using substitution or integration by parts.

In general, all examples need to be covered. Exercises are included at the end of Chapters 1 and 2 in the module. They may be assigned as homework or used as quiz material. Students should read the module and understand the examples. Refer to Outline for Integration included in the module:

Section 6.1 covers I-A-1 and 2.

Section 6.4 covers I-B-1 and 2, II-A.

Section 7.2 covers II-B.

The flowchart included in the module should be used with all examples, as appropriate. I want students to have the flowchart in their head as they work problems and tests.

Posttest Exam I will be handed out later - it will be administered following completion of Section 7.2 in the text. Posttest Exam II will also be handed out later - it will be administered approximately one to two weeks before the end of the semester.

All exams will be graded by the researcher. If you wish to use one of the Posttest exams as a quiz, the scoring system could be adapted for such use.

SECTION 6.1 - ANTIDERIVATIVE (EXPERIMENTAL GROUP)

Objectives:

The student should be able to:
1. Give the notation for and the definition of the antiderivative.
2. Calculate antiderivatives using the *"Basic Rules of Integration."*
3. Solve an application problem involving integrals.
4. Recognize that an integral can be simplified by breaking it up into sums, and then solving it.
5. Recognize that an integral in the form of a rational function can sometimes be simplified by division first.

Lesson Plan

Def. 6.1 - Let f be a function defined on some interval. If F is another function such that

$$F'(x) = f(x)$$

for all x in this interval, we say that F is an antiderivative of f.

If F is an antiderivative of f, than F + c, where c is a constant, is also an antiderivative of f. It is also true that any antiderivative G of f can be written as G(x) = F(x) + c. We call F(x) + c the general antiderivative of f.

If F is a differentiable function, then the differential of F(x) + c is

$$dF = F'(x)\,dx.$$

Introduce the symbol $\int$ (integral sign), which represents the inverse of the differential, that is,

$$dF = F(x) + c.$$

Then

$$\int f(x)\,dx = F(x) + c.$$

So $\int f(x)\,dx$ denotes all antiderivatives of function f (provided f has an antiderivative).

Basic Rules of Integration

1. $\int k\,dx = kx + c$, *for any constant k.*

2. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$, *for* $n \neq -1$.

3. $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + c.$

4. $\int e^x\,dx = e^x + c.$

5. $\int kf(x)\,dx = k\int f(x)\,dx$, *for constant k.*

6. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$

Introduce Chapter 1 in the module, *"Simplify."*

General Rule - Always check for easy alternatives before beginning any complicated or time-consuming operations.
First, check for *"easy algebraic manipulations."*
1. Break integrals into sums.
2. Reduce rational functions to proper fractions by division.
A rational function is a proper fraction if the degree of the numerator is less than the degree of the denominator.

Examples

*Ex.*1 (*no.*5) $\int x^{\frac{2}{5}}\,dx = \frac{1}{\frac{2}{5}+1}x^{\frac{2}{5}+1} + c$

$$= \frac{5}{7}x^{\frac{7}{5}} + c$$

Ex.2 (*no*.6) $\int z^{-\frac{3}{4}}dz = \frac{1}{-\frac{3}{4}+1} z^{-\frac{3}{4}+1} + c$

$= 4z^{\frac{1}{4}} + c$

Ex.3 (*no*.9) $\int (5x^2 - 3x + 4)\,dx$

Simplify? Yes

Rational Fct? No

Break into sums.

$\int (5x^2 - 3x + 4)\,dx = \int 5x^2 dx - \int 3x\,dx + \int 4\,dx$

$= \frac{5}{3}x^3 - \frac{3}{2}x^2 + 4x + c$

Ex.4 (*no*.11) $\int (x+3)(2x-1)\,dx$

Simplify by multiplying together, then split into sums.

$\int (x+3)(2x-1)\,dx = \int (2x^2 + 5x - 3)\,dx$

$= \int 2x^2 dx + \int 5x\,dx - \int 3\,dx$

$= \frac{2}{3}x^3 + \frac{5}{2}x^2 - 3x + c$

Ex.5 (*no*.22) $\int \frac{x+5}{x}\,dx$

Simplify? Yes

Rational Fct? No

Break into sums, simplify.

$\int \frac{x+5}{x}\,dx = \int (\frac{x}{x} + \frac{5}{x})\,dx$

$= \int 1\,dx + \int \frac{5}{x}\,dx$

$= x + 5\ln x + c$

*Ex.*6 (*no.*24) $\int \frac{x^2+3}{\sqrt{x}}\,dx$

Simplify? Yes

Rational Fct? Yes

Is degree of numerator ≤ degree of denominator? No

Improper fraction so divide (simplify).

Break into sums.

$$\int \frac{x^2+3}{\sqrt{x}}\,dx = \int \left(\frac{x^2}{\sqrt{x}} + \frac{3}{\sqrt{x}}\right)dx$$

$$= \int [x^2 x^{-\frac{1}{2}} + 3x^{-\frac{1}{2}}]\,dx$$

$$= \int [x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}]\,dx$$

$$= \int x^{\frac{3}{2}}\,dx + \int 3x^{-\frac{1}{2}}\,dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + c$$

*Ex.*7 (*no.*32) $MC(q) = 18{,}000 - 50q - .06q^2, 0 \le q \le 150$

fixed costs $850 (*this is the constant*), *what is total cost?*

Simplify? Yes

Rational Fct? No

Break into sums.

$$C(q) = \int MC(q)\,dq = \int (18000 - 50q - .06q^2)\,dq$$

$$= \int 18000\,dq - \int (50q)\,dq - \int (.06q^2)\,dq$$

$$= 18000q - 25q^2 - .02q^3 + 850$$

$$C(30) = 18000(30) - 25(30)^2 - .02(30)^3 + 850$$

$$= \$517{,}810.$$

SECTION 6.2 - THE DEFINITE INTEGRAL (EXPERIMENTAL GROUP)

Objectives:

The student should be able to:

1. Recognize a definite integral using the definition
2. Calculate a definite integral using formula 6.6 and theorem 6.1.
3. Solve an application problem with a definite integral.

Lesson Plan

Def. 6.2 - If f is a continuous function and F is an antiderivative of f, then the change in F, when the independent variable changes from x=a to x=b. is denoted by

$$\int_a^b f(x)\,dx$$

and called the definite integral of f from a to b. The values a and b are called limits of integration.

Let $F(x)]_a^b$ denote this difference so that

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

Examples

*Ex.*1 (*no*.3) $\int_{2}^{10}(2x-1)\,dx$

Simplify? Yes
Rational Fct? No
Break into sums.

$$\int_{2}^{10}(2x-1)\,dx = \int_{2}^{10}(2x)\,dx - \int_{2}^{10}(1)\,dx$$
$$= 2\int_{2}^{10}(x)\,dx - \int_{2}^{10}(1)\,dx$$
$$= 2\left[\frac{1}{1+1}x^{1+1}\right]_{2}^{10} - \left[\frac{1}{0+1}x^{0+1}\right]_{2}^{10}$$
$$= 2\left[\frac{1}{2}x^{2}\right]_{2}^{10} - [x]_{2}^{10}$$
$$= 2[50-2] - [10-2]$$
$$= 96 - 8$$
$$= 88$$

*Ex.*2 (*no*.9) $\int_{0}^{3}(y^2-6y+4)\,dy$

Simplify? Yes
Rational Fct? No
Break into sums.

$$\int_{0}^{3}(y^2-6y+4)\,dy = \int_{0}^{3}(y^2)\,dy - 6\int_{0}^{3}(y)\,dy + 4\int_{0}^{3}(1)\,dy$$
$$= \left[\frac{1}{2+1}y^{2+1}\right]_{0}^{3} - 6\left[\frac{1}{1+1}y^{1+1}\right]_{0}^{3} + 4[y]_{0}^{3}$$
$$= \left[\frac{1}{3}y^{3}\right]_{0}^{3} - 6\left[\frac{1}{2}y^{2}\right]_{0}^{3} + 4[y]_{0}^{3}$$
$$= \left[\left(\frac{1}{3}\right)(27) - \left(\frac{1}{3}\right)(0)\right] - 6\left[\left(\frac{1}{2}\right)(9) - \left(\frac{1}{2}\right)(0)\right] + 4[3-0]$$
$$= 9 - 27 + 12$$
$$= -6$$

Theorem 6.1 Properties of the Definite Integral:

1. $\int_a^a f(x)\,dx = 0$

2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *, for any c with a < c < b.*

Examples

*Ex.*3 (*no.*15) $\int_1^5 \frac{1}{x^2}\,dx$

Solution immediate if rewrite as negative exponent.

$$\int_1^5 \frac{1}{x^2}\,dx = \int_1^5 x^{-2}\,dx = [\frac{1}{-2+1}x^{-2+1}]_1^5$$

$$= [-x^{-1}]_1^5$$

$$= [-\frac{1}{x}]_1^5$$

$$= [-\frac{1}{5} - (-1)]$$

$$= \frac{4}{5}$$

Ex.4 (*no*.26) $\int_{-2}^{-1} \frac{5x^2+x+3}{x^2}\,dx$

Simplify? Yes

Rational Fct? Yes

Proper Fraction? Yes

Divide, Break into sums.

$$\int_{-2}^{-1} \frac{5x^2+x+3}{x^2}\,dx = \int_{-2}^{-1} \left(5+\frac{1}{x}+\frac{3}{x^2}\right) dx$$

$$= 5\int_{-2}^{-1} dx + \int_{-2}^{-1} x^{-1}\,dx + 3\int_{-2}^{-1} x^{-2}\,dx$$

$$= 5\,[x]_{-2}^{-1} + [\ln x\,]_{-2}^{-1} + 3\,[-x^{-1}]_{-2}^{-1}$$

$$= 5(-1+2) + (\ln 1 - \ln 2) + 3\left(1-\frac{1}{2}\right)$$

$$= 5 + (0 - .693) + 1.5$$

$$= 5.807$$

Ex.5 (no.30) $\int_1^{25} \frac{x^2+3x-1}{\sqrt{x}} dx$

Simplify? Yes
Rational Fct? Yes
Proper fraction? Yes
Reduce, go A.
Simplify? Yes
Rational Fct? No
Break into sums.

$$\int_1^{25} \frac{x^2+3x-1}{\sqrt{x}} dx = \int_1^{25} (x^2+3x-1)(x^{-\frac{1}{2}}) dx$$

$$= \int_1^{25} (x^{\frac{3}{2}} + 3x^{\frac{1}{2}} - x^{-\frac{1}{2}}) dx$$

$$= \int_1^{25} x^{\frac{3}{2}} dx + 3\int_1^{25} x^{\frac{1}{2}} dx - \int_1^{25} x^{-\frac{1}{2}} dx$$

$$= \frac{2}{5} x^{\frac{5}{2}} \Big]_1^{25} + 3\Big[\frac{2}{3} x^{\frac{3}{2}}\Big]_1^{25} - \Big[2x^{\frac{1}{2}}\Big]_1^{25}$$

$$= (\frac{2}{5})(3125) - (\frac{2}{5})(1) + (2)(125) - 2(1) - 2(5) + 2(1)$$

$$= 1250 - \frac{2}{5} + 250 - 2 - 10 + 2$$

$$= 1489.6$$

Ex.6 (no.33) $\int_{90}^{100} (300 - 1.4q - 6\sqrt{q}\, dq$

Simplify? Yes
Rational Fct? No
Break into sums.

$$MC = 300 - 1.4q - 6\sqrt{q},\ 50 \le q \le 120$$

$$\int_{90}^{100} 300 - 1.4q - 6\sqrt{q}\, dq = [300q - .7q^2 - 4q^{\frac{3}{2}}]_{90}^{100}$$

$$= 30000 - 7000 - 4000 - 27000 + 5670 + 3415.26$$

$$= \$1085.26$$

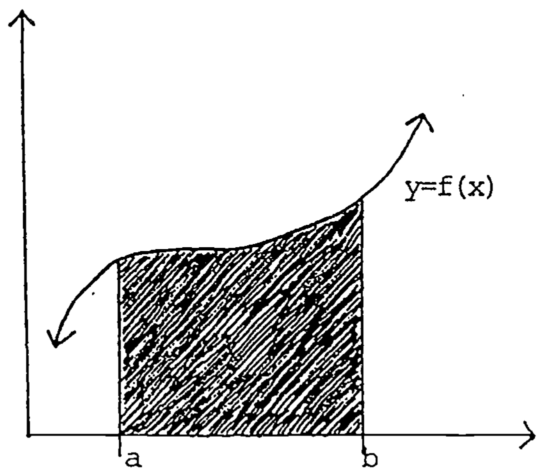
SECTION 6.3 AREA (EXPERIMENTAL GROUP)

Objectives:
The student should be able to:
1. Find the area of a bounded, single function region using the definite integral.
2. Find the area of a region bounded by two functions using the definite integral.

Lesson Plan

Theorem 6.2 - If f is positive and continuous on [a,b], then the area of the region bounded by y = f(x), x=a, x=b, and the x-axis is given by the definite integral

$$\int_a^b f(x)\,dx$$



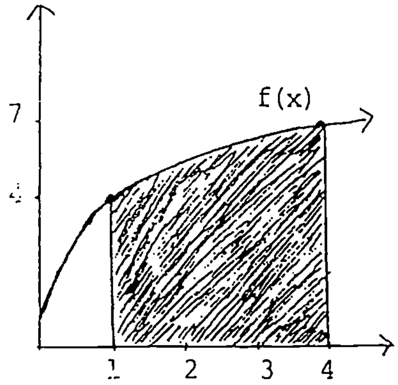
Examples

*Ex.*1 (*no*.3) $f(x) = 1 + 3\sqrt{x}$; $a = 1, b = 4$

Simplify? Yes

Rational Fct? No

Break into sums.



$$\int_1^4 (1 + 3\sqrt{x})\,dx = [x + 2x^{\frac{3}{2}}]_1^4$$

$$= [4 + 2(8)] - [1 + 2(1)]$$

$$= 20 - 3$$

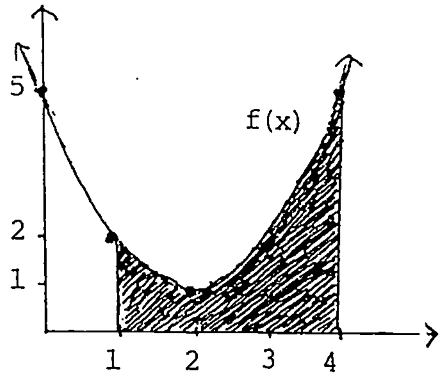
$$= 17$$

*Ex.*2 (*no*.5) $f(x) = x^2 - 4x + 5$; $a = 1, b = 4$

Simplify? Yes

Rational Fct? No

Break into sums.



$$-\frac{b}{2a} = \frac{-(-4)}{2(1)} = 2$$

$f(2) = 4 - 8 + 5 = 1$

vertex = (2,1)

$$\int_1^4 (x^2 - 4x + 5)\,dx = [\frac{1}{3}x^3 - 2x^2 + 5x]_1^4$$

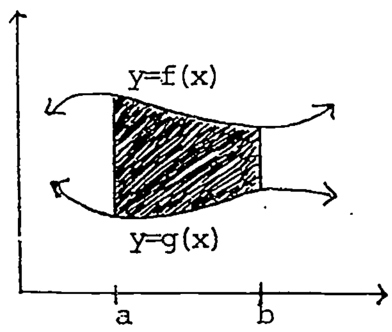
$$= [(\frac{1}{3})(4)^3 - (2)(4)^2 + 5(4)] - [(\frac{1}{3})(1)^3 - 2(1)^2 + 5(1)]$$

$$= \frac{64}{3} - 32 + 20 - \frac{1}{3} + 2 - 5$$

$$= 6$$

Theorem 6.3 - If f and g are continuous functions on [a,b], and if $f(x) \geq g(x)$ for all x in [a,b], then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, and between $x=a$ and $x=b$ is given by the definite integral

$$\int_a^b [f(x) - g(x)]\, dx.$$



Examples

*Ex.*3 (*no.*11) $f(x) = x^2 - x + 5; g(x) = \sqrt{x}; a = 1, b = 4$

Simplify? Yes

Rational Fct? No

Break into sums.

$f(1) = 5,\ g(1) = 1,\ so\ f(1) \geq g(1)$

$f(4) = 17,\ g(4) = 2,\ so\ f(4) \geq g(4)$

$$\int_1^4 [(x^2 - x + 5) - (\sqrt{x})]\,dx = [\frac{1}{3}x^3 - \frac{1}{2}x^2 + 5x - \frac{2}{3}x^{\frac{3}{2}}]_1^4$$

$$= \frac{64}{3} - 8 + 20 - \frac{16}{3} - \frac{1}{3} + \frac{1}{2} - 5 + \frac{2}{3}$$

$$= \frac{143}{6}$$

*Ex.*4 (*no.*21) $f(x) = \frac{1}{x}, g(x) = \frac{5}{4} - \frac{1}{4}x$

Set $f(x) = g(x)$ *for intersection points*

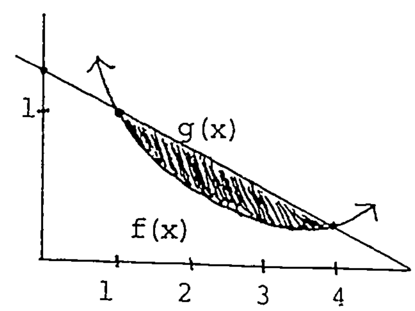
$$\frac{1}{x} = \frac{5}{4} - \frac{x}{4}$$

$$4 = 5x - x^2$$

$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1)\ 0$$

$$x = 4\quad x = 1$$



$$\int_1^4 [\frac{5}{4} - \frac{1}{4}x - \frac{1}{x}]\,dx$$

Simplify? Yes
Rational Fct? No
Break into sums.

$$\int_1^4 [\frac{5}{4} - \frac{1}{4}x - \frac{1}{x}]\,dx = [\frac{5}{4}x - \frac{1}{8}x^2 - \ln x]_1^4$$

$$= [(\frac{5}{4})(4) - (\frac{1}{8})(4)^2 - \ln 4] - [(\frac{5}{4})(1) - (\frac{1}{8})(1)^2 - \ln 1]$$

$$= 5 - 2 - \ln 4 - \frac{5}{4} + \frac{1}{8} + 0$$

$$= \frac{15}{8} - \ln 4$$

SECTION 6.4 - THE METHOD OF SUBSTITUTION
(EXPERIMENTAL GROUP)

Objectives:

The student should be able to:
1. Recognize an integral with a function of a function and, substituting for the *"inside"* function, solve a definite or indefinite integral.
2. Recognize when to substitute for a denominator or complicated function and, using the method of substitution, solve a definite or indefinite integral.

Lesson Plan

Definition - Making a substitution in the integrand to aid in finding an antiderivative is called the method of substitution.

Guidelines to use in looking for substitutions:
1. Does the integrand contain a function of a function?
2. Does the integrand contain a complicated function, particularly in the denominator of a fraction?
Note: In general, a substitution $u = f(x)$ will only help if you can find the term $du = f'(x)\,dx$ somewhere in the integral.

Obvious substitutions:
1. *"Inside"* functions.
2. *"Complicated"* terms and denominators.

Examples

*Ex.*1 (*no*.3) $\int (2x+5)^3 dx$

Simplify? Yes

Rational Fct? No

Break into sums? No

Does integral contain fct. of fct.? Yes

Inside fct so

let $u = 2x+5,\ du = 2\,dx$

$$\int (2x+5)^3 dx = \int (2x+5)^3 (\frac{1}{2})(2) dx$$

$$= \frac{1}{2} \int u^3 du$$

$$= \frac{1}{2}(\frac{1}{4}u^4) + c$$

$$= \frac{1}{8}(2x+5)^4 + c$$

*Ex.*2 (*no*.8) $\int s^5 (4-3s^6) ds$

Simplify? Yes

Rational Fct? No

Break into sums? No

Fct. of Fct.? Yes

Inside function so

let $u = 4-3s^6,\ du = -18s^5 ds$

$$\int s^5 (4-3s^6) ds = \int (-\frac{1}{18})(-18) s^5 (4-3s^6) ds$$

$$= -\frac{1}{18} \int u\, du$$

$$= -\frac{1}{18}(\frac{1}{2}u^2) + C$$

$$= -\frac{1}{36}(4-3s^5)^2 + C$$

*Ex.*4 (*no*.36) $\int \frac{\sqrt{\ln x}}{x}\,dx$

Simplify? Yes
Rational Fct? Yes
Degreenum. ≤ degree denom.? Yes
Break into sums? No
Fct. of Fct.? Yes
Inside function so

$$\text{let } u = \ln x,\ du = \frac{1}{x}\,dx$$

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \frac{1}{x}\sqrt{\ln x}\,dx$$

$$= \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}(\ln x)^{\frac{3}{2}} + C$$

*Ex.*5 (*no*.37) $\int \frac{\ln^4(3-x)}{3-x}\,dx$

Simplify? Yes
Rational Fct? Yes
Degree num. ≤ degree denom.? Yes
Break into sums? No
Fct. of Fct.? Yes
Inside function so

$$\text{let } u = \ln(3-x),\ du = \frac{1}{3-x}\,dx$$

$$\int \frac{\ln^4(3-x)}{3-x}\,dx = \int u^4\,du$$

$$= \frac{1}{5}u^5 + C$$

$$= \frac{1}{5}\ln^5(3-x) + C$$

Ex.6 (no.40) $\int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$

Simplify? Yes

Rational Fct.? Yes

Degree num. ≤ degree denom.? Yes

Break into sums? No

Fct. of Fct.? No

Complicated fct. for denom. so

let $u = e^{x} + e^{-x}$, $du = e^{x} - e^{-x}dx$

$$\int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx = \int \frac{du}{u}$$
$$= \int \frac{1}{u}du$$
$$= \ln u + C$$
$$= \ln e^{x}+e^{-x} + C$$

Ex.7 (no.49) $\int_0^2 \frac{dx}{8x+1}$

Simplify? No

Classify? Yes

Rational Fct? Yes

Form $\frac{r}{ax+b}$ *so*

let $u = 8x+1$, $du = 8dx$

$$\int_0^2 \frac{dx}{8x+1} = \int_0^2 (\frac{1}{8x+1})(\frac{1}{8})(8)\,dx$$
$$= \frac{1}{8}\int \frac{1}{u}du$$
$$= \frac{1}{8}[\ln u\,]$$
$$= \frac{1}{8}[\ln 8x+1\,]_0^2$$
$$= \frac{1}{8}[\ln 17 - \ln 1]$$
$$= \frac{1}{8}\ln 17$$

A basic rational function is a proper function of the form

$$\frac{r}{ax+b}, \frac{r}{ax+b^n}, \frac{rx+s}{ax^2+bx+c}.$$

The first two are easy to integrate (let u = ax+b); the third form will not be discussed in this course.

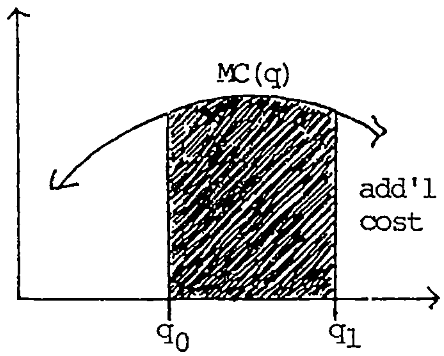
SECTION 6.5 - APPLICATIONS OF THE DEFINITE INTEGRAL
(EXPERIMENTAL GROUP)

Objectives:

The student should be able to:
1. Find increased cost given marginal cost.
2. Find increased revenue given marginal revenue.
3. Find the equilibrium point, producer's surplus, and consumer's surplus given supply and demand functions.
4. Find the mean value of a function over a given interval.

Lesson Plan

Increased Cost - area of a region bounded by marginal cost function.



(Figure 6.9)

Example

Ex. (*no*.4) $MC = 10 - .08q + .006q^2,\ 0 \le q \le 150$

increase q, 100 – 120, find increased cost.

Simplify? Yes

Rational Fct? No

Break into sums.

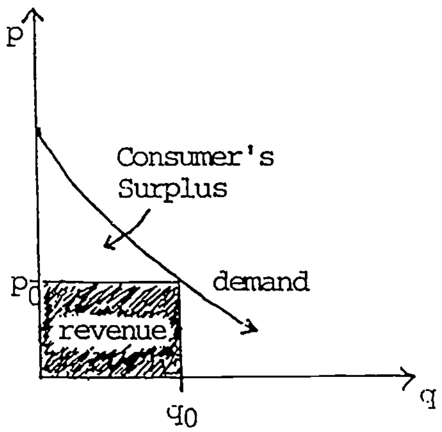
$$\int_{100}^{120} MC(q) = \int_{100}^{120} (10 - .08q + .006q^2)\,dq$$

$$= [10q - .04q^2 + .002q^3]_{100}^{120}$$

$$= 4080 - 2600$$

$$= \$1480$$

Consumer's Surplus - area of region below demand function but above horizontal line $p = p_0$, representing money not spent by consumers who would have been willing to pay a price higher than p_0 for the product.



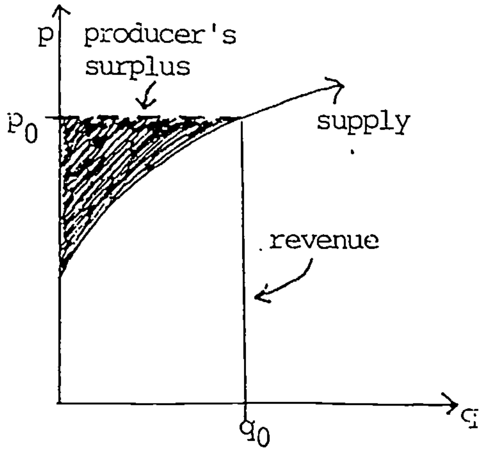
(Figure 6.10)

$$CS = \int_{0}^{q_0} [f(q) - p_0]\,dq$$

where p_0 is selling price, q_0 is demand level,

$p = f(q)$ is demand function.

Producer's Surplus - area of region above supply curve and below horizontal line $p=p_0$, representing money that suppliers would not have received if demand had been less than q_0.



$$PS = \int_0^{q_0} [p_0 - f(q)]\,dq$$

where $p = f(q)$ is supply function,
q_0 is demand level.

Examples

Consumer's Surplus, Producer's Surplus.

$$2.8 + .5q - .002q^2 = 25 - .36q + .002q^2$$
$$0 = .004q^2 - .86q + 22.2$$
$$q = \frac{-(-.86) \pm \sqrt{(-.86)^2 - 4(.004)(22.2)}}{(2)(.004)}$$
$$= \frac{.86 \pm \sqrt{.7396 - .3552}}{.008}$$
$$= \frac{.86 \pm .62}{.008}$$

$q = 185$, $q = 30$, *but* $0 \le q \le 90$ *so* $q = 30$.

for $q = 30$,

$$p(30) = 2.8 + .5(30) - .002(30)^2 = 16 = p$$

$$PS = \int_0^{30} [16 - (2.8 + .5q - .002q^2)]\, dq$$

Simplify? Yes
Rational Fct? No
Break into sums.

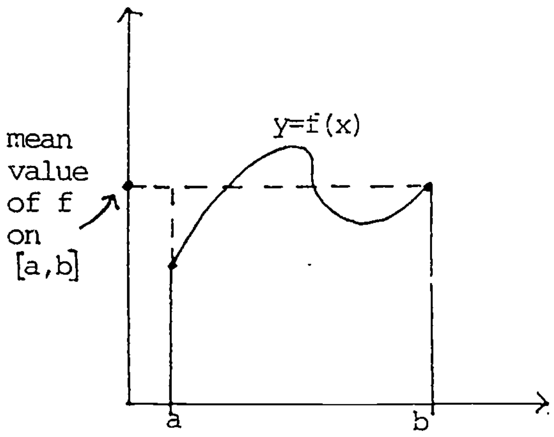
$$PS = \int_0^{30} [16 - (2.8 + .5q - .002q^2)]\, dq$$
$$= \int_0^{30} (13.2 - .5q + .002q^2)\, dq$$
$$= [13.2q - .25q^2 + \frac{.002}{3}q^3]_0^{30}$$
$$= (189 - 0)$$
$$= \$189$$

$$CS = \int_0^{30} [(25 - .36q + .002q^2) - 16]\, dq$$

Simplify? Yes
Rational Fct? No
Break into sums.

$$CS = \int_0^{30} [(25 - .36q + .002q^2) - 16]\, dq$$
$$= \int_0^{30} (.002q^2 - .36q + 9)\, dq$$
$$= [\frac{.002}{3}q^3 - .18q^2 + 9q]_0^{30}$$
$$= \$126$$

Mean Value - the height of a rectangle whose area equals that of the region bounded above by y = f(x) between x=a and x=b.



$$MV = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Examples

*Ex.*3 (*no.*17) $g(x) = x^3$ *on* $[-1,1]$

Solution immediate.

$$MV = \frac{1}{1-(-1)}\int_{-1}^{1} x^3\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^3\,dx$$

$$= \frac{1}{2}\left[\frac{1}{4}x^4\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{4}-\frac{1}{4}\right)$$

$$= 0$$

*Ex.*4 (*no.*27) $V = 50{,}000e^{-.3t}, 0 \le t \le 10$

Solution immediate.

$$MV = \frac{1}{10-0}\int_0^{10} 50{,}000e^{-.3t}\,dt$$

$$= \frac{1}{10}[(50{,}000)(e^{-.3t})(\frac{1}{-.3})]_0^{10}$$

$$= \frac{1}{10}[-166666.7e^{-.3t}]_0^{10}$$

$$= \frac{1}{10}(-8297.85 + 166666.7)$$

$$= \$15836.89$$

SECTION 7.1 - RIEMANN SUMS AND NUMERICAL INTEGRATION
(EXPERIMENTAL GROUP)

Objectives:

The student should be able to:
1. Evaluate the Riemann sum for a function over some interval given values of N.
2. Estimate the value of a given definite integral using the midpoint rule.
3. Use the trapezoidal rule to estimate the value of a given integral.
4. Use the trapezoidal rule to estimate a definite integral given a set of tabular data.

Lesson Plan

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

when f is continuous on [a,b] and F is an antiderivative of f.

Riemann Sum - approximates the area under a curve by partitioning the interval into subintervals, replacing the graph of the function with an approximating horizontal segment over each subinterval, and summing the areas of the resulting rectangles. To construct N subdivisions of equal width Δx.

given

$$\int_a^b f(x)\,dx, \textit{ take } \Delta x = \frac{b-a}{N}.$$

For the height of each rectangle use f(c) for some c in the subinterval. The area of the rectangle is then $f(c) \cdot \Delta x$. If c_1 is selected in the first subinterval, c_2 in the second, and so on, the sum of these areas

$$S = f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_N)\Delta x$$

is an approximation for $\int_a^b f(x)\,dx$. *See figure* 7.2.

Examples

*Ex.*1 (*no*.3) $f(x) = 3x^2 + 1$ *on* $[0,4]$; $N = 5$,

$$c_1 = 0,\ c_2 = 1,\ c_3 = 2,\ c_4 = 3,\ c_5 = 3.8.$$

$$(estimate \int_0^4 (3x^2+1)\,dx).$$

$$\Delta x = \frac{4-0}{5} = .8$$

$$f(c_1) = f(0) = 1; f(c_2) = f(1) = 4$$

$$f(c_3) = f(2) = 13; f(c_4) = f(3) = 28$$

$$f(c_5) = f(3.8) = 44.3$$

$$S = (1)(.8) + (4)(.8) + (13)(.8) + (28)(.8) + (44.3)(.8)$$
$$= 72.24$$

$$\int_0^4 (3x^2+1)\,dx$$

Simplify? Yes
Rational Fct? No
Break into sums.

$$\int_0^4 (3x^2+1)\,dx = [x^3 + x]_0^4$$
$$= [(4)^3 + 4] - 0$$
$$= 68$$

Midpoint Rule - specialized Riemann sum used to approximate definite integrals. Each c_i is selected as the midpoint of the subinterval.

Ex.2 (no.7) $f(x) = \frac{x}{x^2+1}$ *on* $[1,4]; N=3,$

$$c_1 = 1.1, c_2 = 2.5, c_3 = 3.6$$

$$\Delta x = \frac{4-1}{3} = 1$$

$$f(c_1) = .4977; f(c_2) = .3448; f(c_3) = .2579$$

$$S = [.4977 + .3448 + .2579](1) = 1.1$$

$$\int_1^4 \frac{x}{x^2+1}\,dx$$

Simplify? Yes
Rational Fct? Yes
Proper fraction? Yes
Break into sums? No
Fct of a Fct? Yes
Complicated fct for denom. so
let $u = x^2+1$, $du = 2x\,dx$

$$\int_1^4 \frac{x}{x^2+1}\,dx = \int_1^4 \frac{1}{2}(\frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}[\ln u]$$

$$= \frac{1}{2}[\ln x^2+1]_1^4$$

$$= \frac{1}{2}[\ln 17 - \ln 2]$$

$$= \frac{1}{2}\ln\frac{17}{2}$$

$$= \frac{1}{2}\ln 8.5$$

Example

$$Ex.3\ (no.13)\ \int_1^3 \frac{1}{x}\,dx, N=5$$

$$\Delta x = \frac{3-1}{5} = \frac{2}{5} = .4$$

subintervals are

$$[1,1.4], [1.4,1.8], [1.8,2.2], [2.2,2.6], [2.6,3.0]$$

$$c_1 = 1.2, c_2 = 1.6, c_3 = 2.0, c_4 = 2.4, c_5 = 2.8$$

$$\begin{aligned} S &= [\frac{1}{1.2} + \frac{1}{1.6} + \frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.8}](.4) \\ &= (2.7321)(.4) \\ &= 1.0929 \end{aligned}$$

Trapezoidal Rule - approximates a definite integral using the sum of areas of trapezoids rather than rectangles (see figure 7.5). Use right endpoint values to form

$$S_R = [f(c_1) + f(c_2) + \ldots + f(c_{N-1}) + f(b)]\Delta x$$

Use left endpoint values to form

$$S_L = [f(a) + f(c_1) + f(c_2) + \ldots + f(c_{N-1})]\Delta x.$$

Then the Trapezoidal Rule is:

$$\begin{aligned} T &= \frac{S_L + S_R}{2} \\ &= \frac{\Delta x}{2}[f(a) + 2(f(c_1) + f(c_2) + \ldots + f(c_{N-1})) + f(b)] \end{aligned}$$

Examples

$$Ex.4\ (no.19)\ \int_0^2 \sqrt{1+x^3}\,dx\ for\ N=5$$

$$\Delta x = \frac{2-0}{5} = \frac{2}{5} = .4$$

subintervals are

$$[0,.4],[.4,.8],[.8,1.2],[1.2,1.6],[1.6,2.0]$$

$$t_5 = .2(1+2(1.03+1.23+1.65+2.26)+3)$$

$$= 3.268$$

$$for\ N = 10,\ \Delta x = \frac{2-0}{10} = \frac{1}{5} = .2$$

subintervalsare

$$[0,.2],[.2,.4],[.4,.6],[.6,.8],[.8,1.0],$$

$$[1.0,1.2],[1.2,1.4],[1.4,1.6],[1.6,1.8],[1.8,2.0]$$

$$T_{10} = .1(1+2(1+1.03+1.1+1.23+1.4+1.7+1.9+2.3+2.6)+3)$$

$$= .1(1+2(14.26)+3)$$

$$= 3.252$$

$$Ex.5\ (no.21)\ \ R(100) - R(50) = \int_{50}^{100} MR(q)\,dq$$

$$\Delta q = 10, N = 5$$

$$\int_{50}^{100} MR(q)\,dq = \frac{10}{2}[MR(50)+2(MR(60)+MR(70)+MR(80)+MR(90))+MR(100)]$$

$$= 5[20+2(15+12+8+4)+2]$$

$$= 5(20+78+2)$$

$$= 500$$

SECTION 7.2 - INTEGRATION TECHNIQUES
(EXPERIMENTAL GROUP)

Objectives:

The student will be able to:
1. Recognize that a product suggests use of integration by parts.
2. Find an antiderivative using integration by parts.
3. Use Table 7.1 to find antiderivative.

Lesson Plan

Integration by parts - based on the product rule for derivatives.

$$\int u\,dv = uv - \int v\,du$$

This is useful if the integrand is a product, especially a product of dissimilar functions. We separate the integral into two parts and call one "*u*" and the other "*dv*." The goal is to choose "*u*" and "*dv*" such that the term $\int v\,du$ is easier to solve than the original problem.

Examples

*Ex.*1 (*no*.8) $\int \sqrt{x} \ln x dx$

$$let\, u = \ln x,\ dv = \sqrt{x} dx$$

$$then\ du = \frac{1}{x} dx,\ v = \int \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}}$$

$$\int \sqrt{x} \ln x dx = \frac{2}{3} x^{\frac{3}{2}} \ln x - \int \frac{2}{3} x^{\frac{3}{2}} x^{-1} dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int x^{\frac{1}{2}} dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} (\frac{2}{3}) x^{\frac{3}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{4}{9} x^{\frac{3}{2}} + C$$

*Ex.*2 (*no*.9) $\int x(2x+1)^4 dx$

$$let\ u = x,\ dv = (2x+1)^4 dx$$

$$then\ du = dx,\ v = \int (2x+1)^4 dx = \frac{(2x+1)^5}{5} (\frac{1}{2})$$

$$\int x(2x+1)^4 dx = x[(\frac{1}{2}) \frac{(2x+1)^5}{5}] - \int (\frac{1}{10} (2x+1)^5) dx$$

$$= \frac{1}{10} x(2x+1)^5 - \frac{1}{10} [\frac{(2x+1)^6}{6} (\frac{1}{2})] + C$$

$$= \frac{1}{10} x(2x+1)^5 - \frac{1}{120} (2x+1)^6 + C$$

Introduce List of Antiderivatives - Table 7.1.
It is usually necessary to make a substitution or a change of variable to put the given integrand into one of the forms found in the table.

Examples

*Ex.*3 (*no.*17) $\int x^2 4^{6-x^3} dx$

$$use\ no.\ 4, \int b^u du = \frac{1}{\ln b} b^u + C$$

$$let\ u = 6 - x^3, du = -3x^2 dx$$

$$\int x^2 4^{6-x^3} dx = \int 4^{6-x^3}(-\frac{1}{3})(-3x^2 dx)$$

$$= -\frac{1}{3}\int 4^u du$$

$$= (-\frac{1}{3})[\frac{1}{\ln 4}(4^{6-x^3})] + C$$

*Ex.*4 (*no.*18) $\int \frac{dx}{25x^2-9} = \int \frac{dx}{(5x)^2-(3)^2}$

$$use\ no.\ 14, \int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln \frac{u-a}{u+a} + C$$

$$let\ u = 5x, du = 5\,dx, a = 3$$

$$\int \frac{dx}{25x^2-9} = \int (\frac{1}{5})(\frac{5\,dx}{(5x)^2-(3)^2})$$

$$= \frac{1}{5}[\frac{1}{2(3)}\ln \frac{5x-3}{5x+3}] + C$$

$$= \frac{1}{30}\ln \frac{5x-3}{5x+3} + C$$

Ex.5 (no.27) $\int_{-1}^{2} x e^{\frac{x}{2}} dx$

use no.5, $\int u e^{u} du = u e^{u} - e^{u} + C$

let $u = \frac{1}{2}x, du = \frac{1}{2}dx$

$$\int_{-1}^{2} x e^{\frac{x}{2}} dx = \int_{-1}^{2} (2)(\frac{1}{2}x) e^{\frac{x}{2}} (2)(\frac{1}{2}dx)$$

$$= \int 4 u e^{u} du$$

$$= 4\int u e^{u} du$$

$$= 4[u e^{u} - e^{u}]$$

$$= 4[\frac{x}{2} e^{\frac{x}{2}} - e^{\frac{x}{2}}]_{-1}^{2}$$

$$= 4[(\frac{2}{2} e^{\frac{2}{2}} - e^{\frac{2}{2}}) - (-\frac{1}{2} e^{-\frac{1}{2}} - e^{-\frac{1}{2}})]$$

$$= 4(\frac{3}{2} e^{-\frac{1}{2}})$$

$$= 6 e^{-\frac{1}{2}}$$

APPENDIX C

PRETEST

INSTRUCTIONS FOR ADMINISTERING PRETEST

Please write on the board and read to your students the following instructions:

Time Limit: 15 minutes exactly

Part I: (Do this first). State the technique you will use. If substitution, indicate your choice for u; if integration by parts, indicate choices for u and dv.

Part II: Evaluate the integral using the technique chosen. If you decide to use a different technique, cross through the original and add the new one. DO NOT ERASE.

NAME______________________

PRETEST EXAM - INDEFINITE INTEGRATION

You will have 15 minutes to complete this test.

Part I:

State the technique you will use to solve each integral (substitution, integration by parts, etc.). Once you have chosen a technique for each problem, go to Part II.

1. $\int \frac{3}{(2z-1)^3} dz$

2. $\int \frac{\ln^2(x+4)}{x+4} dx$

3. $\int \sqrt{x} \ln x \, dx$

4. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

5. $\int \frac{x^2+3}{\sqrt{x}} dx$

Part II: Evaluate each integral using the techniques chosen above. If, in evaluating the integral, you decide to use a different technique, cross through the original and add the new technique. DO NOT ERASE.

1. $\int \frac{3}{(2z-1)^3} dz$

2. $\int \frac{\ln^2(x+4)}{x+4}\,dx$

3. $\int \sqrt{x}\ln x\,dx$

4. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

5. $\int \frac{x^2+3}{\sqrt{x}}\,dx$

APPENDIX D

POSTTEST I

INSTRUCTIONS FOR ADMINISTERING POSTTEST I

It is crucial to the success of my study that students understand and follow instructions for this posttest. Explain that I am more interested in the technique chosen and substitutions used than the actual computation or solution to the problem. The majority of points will be given in Part I, with Part II a secondary source. Do not advise students of the exam prior to giving it.

Please write on the board and read to your students the following instructions:

Time Limit: 15 minutes exactly

Part I: (Do this first). State the technique you will use. If substitution, indicate your choice for u; if integration by parts, indicate choices for u and dv.

Part II: Evaluate the integral using the technique chosen. If you decide to use a different technique, cross through the original and add the new one. DO NOT ERASE.

NAME________________________

POSTTEST EXAM I - INDEFINITE INTEGRATION

You will have 15 minutes to complete this test.

Part I: State the technique you will use to solve each integral (substitution, integration by parts, etc.). Once you have chosen a technique for each problem, go to Part II.

1. $\int \frac{3}{(2z-1)^3} dz$

2. $\int \frac{\ln^2(x+4)}{x+4} dx$

3. $\int \sqrt{x} \ln x \, dx$

4. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

5. $\int \frac{x^2+3}{\sqrt{x}} dx$

Part II: Evaluate each integral using the techniques chosen above. If, in evaluating the integral, you decide to use a different technique, cross through the original and add the new technique. DO NOT ERASE.

1. $\int \frac{3}{(2z-1)^3} dz$

2. $\int \frac{\ln^2(x+4)}{x+4}\,dx$

3. $\int \sqrt{x}\ln x\,dx$

4. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

5. $\int \frac{x^2+3}{\sqrt{x}}\,dx$

APPENDIX E

POSTTEST II

INSTRUCTIONS FOR ADMINISTERING POSTTEST II

Again, it is crucial that students understand and follow instructions for this test. I am more interested in the technique chosen and substitutions used than the actual computation or solution to the problem. The majority of points will again by given in Part I, with Part II a secondary source.

Please give this posttest before you start your review. Do not advise students of the exam prior to giving it. I want to measure long-term retention with no preparation.

Time Limit: 15 minutes exactly

Part I: Do this first. State the technique you will use. If substitution, indicate your choice for u; if integration by parts, indicate choices for u and dv.

Part II: Evaluate the integral using the technique chosen. If you decide to use a different technique, cross through the original and add the new one. DO NOT ERASE.

NAME________________________

POSTTEST EXAM II - INDEFINITE INTEGRATION

You will have 15 minutes to complete this test.

Part I: State the technique you will use to solve each integral (substitution, integration by parts, etc.). Once you have chosen a technique for each problem, go to Part II.

1. $\int \frac{3}{(5-x)^2}\,dx$

2. $\int \frac{\ln^3(2x-1)}{2x-1}\,dx$

3. $\int \sqrt{x}\ln x\,dx$

4. $\int \frac{e^{-x}-e^{x}}{e^{x}+e^{-x}}\,dx$

5. $\int \frac{3x^3-2x}{\sqrt{x}}\,dx$

Part II: Evaluate each integral using the technique chosen above. If, in evaluating the integral, you decide to use a different technique, cross through the original and add the new technique. DO NOT ERASE.

1. $\int \frac{3}{(5-x)^2}\,dx$

2. $\int \frac{\ln^3(2x-1)}{2x-1}\,dx$

3. $\int \sqrt{x}\ln x\,dx$

4. $\int \frac{e^{-x}-e^{x}}{e^{x}+e^{-x}}\,dx$

5. $\int \frac{3x^3-2x}{\sqrt{x}}\,dx$

APPENDIX F

INTEGRATION MODULE

An adaptation of
Alan H. Schoenfeld's

INTEGRATION:
Getting It All Together

(Originally published in June, 1977, by UMAP, Newton, MA.)

Adapted by **Linda Kallam**
1992

INTRODUCTION

This booklet provides students with a general procedure for solving problems in integration. Based on observations of *"experts"* working on integrals, the procedure has three steps: **SIMPLIFY**, **CLASSIFY**, and **MODIFY**.

In step 1, **SIMPLIFY**, we try to reduce a problem to one which can be solved by a formula or can be done easily. If this fails to solve the problem we proceed to step 2, **CLASSIFY**. Here we use the form of the integrand to decide which special technique, i.e., integration by parts, substitution, etc., to use on the problem. If we are unable to **CLASSIFY** the integrand, go to step 3, **MODIFY**. There we try to manipulate the integrand into a more familiar or manageable form. We always check for simple alternatives before beginning complicated calculations, and start the process over with step 1 whenever we have succeeded in transforming the integral to something easier.

OUTLINE FOR INTEGRATION

I. Simplify

A. Easy Algebraic Manipulations

1. Break into sums.

2. Reduce rational functions to proper fractions by division.

B. Obvious Substitutions

1. "*Inside*" functions

2. "*Nasty*" or complicated terms and denominators.

II. Classify

A. Rational Functions - If denominator is (ax + b) or $(ax + b)^n$, substitute u = ax + b.

B. Products - Use integration by parts.

III. Modify

A. Problem Similarities

B. Special Manipulations

C. Needs Analysis

CHAPTER 1

SIMPLIFY !

There is one general rule that you should keep in mind whenever you are solving problems:

ALWAYS CHECK FOR EASY ALTERNATIVES BEFORE BEGINNING ANY COMPLICATED OR TIME-CONSUMING OPERATIONS.

As the sample problems below illustrate, it is worth taking a few moments to look for a quick or easy solution to a problem before jumping into a complicated procedure. This is especially true in integration, where a timely observation can save tremendous amounts of work. The two types of SIMPLIFYing operations we will discuss are summarized below.

Step 1:	**SIMPLIFY**
Easy Algebraic Manipulations	Obvious Substitutions

EASY ALGEBRAIC MANIPULATIONS

Some algebraic manipulations are easy enough to use that it's worth considering them automatically before going on to anything else. For example, we almost always break the integral of a sum into a sum of integrals and then integrate term by term. Before doing this, however, we should look for other alternatives. An operation which is more complicated but also worth considering is simplifying rational functions by long division.

We call a rational function (the quotient of two polynomials) a *"proper fraction"* if the degree of the numerator is less than the degree of the denominator. Proper fractions are usually easier to manipulate than others. In sum, we have:

EASY ALGEBRAIC MANIPULATIONS
(1) Break integrals into <u>sums</u> (2) Reduce rational functions to <u>Proper Fractions</u> by division.

SAMPLE PROBLEMS

This sample problem can be **SIMPLIFIED** by an easy algebraic manipulation. Try to solve it before you read the solution, and then compare your method with the solution.

$$\int \frac{x^3+1}{x^2}dx$$

SOLUTION

The integrand in this problem is an *"improper fraction,"* so we should perform a division. The division gives us a quotient of (x) and a remainder of $(1/x^2)$, so we obtain

$$\int \frac{x^3+1}{x^2}\,dx = \int (x+\frac{1}{x^2})\,dx$$

If we break the integrand into sums, the above becomes

$$\int x\,dx + \int \frac{1}{x^2}\,dx = \int x\,dx + \int x^{-2}dx = \frac{1}{2}x^2 - \frac{1}{x} + C$$

"OBVIOUS" SUBSTITUTIONS

Using substitution is one of the most powerful tools we have for **SIMPLIFY**ing and solving integrals. Always look for substitu-tions before trying more complex procedures. There are two guidelines to use in looking for substitutions:

(1) Does the integrand contain a function of a function?

If it does, try a substitution with u as the *"inside"* function. Consider the integral

$$\int \frac{x}{x^2+3} \ln(x^2+3)dx.$$

The term $\ln(x^2 + 3)$ has $x^2 + 3$ as an inside function. Try the substitution $u = x^2 + 3$.

(2) Does the integrand contain a complicated or *"nasty"* function, particularly in the denominator of a fraction?

If so, try a substitution with u as the *"nasty"* function. Consider

$$\int \frac{x}{x^2-9}dx.$$

The denominator isn't particularly *"nasty,"* but it's worth trying the substitution $u = x^2 - 9$. Then $du = 2x\,dx$, and the integral is

$$\frac{1}{2}\int \frac{2x\,dx}{x^2-9} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + C = \frac{1}{2}\ln x^2-9 + C.$$

NOTE: If the problem were $\int x^2 - 9\,dx$, the substitution $u = x^2 - 9$ would not have helped. In general, a substitution $u = f(x)$ will only help if you can find the term $du = f'(x)dx$ somewhere in the integral. If you try a substitution and it looks like you're getting involved in a complicated procedure, stop to consider other alternatives. The procedures of chapter 1 are designed to help **SIMPLIFY** and solve an integral rapidly. You should explore all simple alternatives before trying anything complicated. If need be, you can always return to a complicated substitution later.

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OBVIOUS SUBSTITUTIONS
(1) "*Inside*" functions (2) "*Nasty*" terms and denominators

SAMPLE PROBLEMS

1. $\int \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} dx$

2. One of the following two integrals is much easier to solve than the other. Decide which it is, and solve it.

(*a*) $\int x^{3}(1+x^{4})^{5} dx$ (*b*) $\int (1+x^{4})^{5} dx$

SOLUTIONS

1. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

We will work this problem using all the methods of this chapter, to illustrate how you would think about this problem if you didn't know where it came from.

As a first step, look for algebraic simplifications. The numerator is a sum, so you might consider breaking the integral up into

$$\int \frac{e^x}{e^x - e^{-x}} dx + \int \frac{e^{-x}}{e^x - e^{-x}} dx.$$

This doesn't seem to help, so look for substitutions. You might be tempted to try the substitution $u = e^x$ at first, since all the terms in the integral are expressed in terms of e^x. But $du = e^x\,dx$, which is not in the integral. For that reason, there is no need to explore the substitution further now. If necessary, you can return to it later.

Finally, you might try a substitution for the denominator, $u = (e^x - e^{-x})$. This gives $du = (e^x + e^{-x})\,dx$, which does appear in the integral. From here on the problem is easy. We have

$$\int \frac{1}{e^x - e^{-x}} [(e^x + e^{-x})\,dx] = \int \frac{1}{u} du = \ln u + C = \ln \frac{e^x + e^{-x}}{e^x - e^{-x}} + C.$$

2. One of the following two integrals is much easier to solve than the other. Decide which it is, and solve it.

$$(a)\ \int x^3(1+x^4)^5 dx \qquad (b)\ \int (1+x^4)^5 dx$$

As always, start working on a problem by looking for algebraic simplifications. In both parts (a) and (b) of this problem, you can multiply $(1 + x^4)$ by itself five times, and then integrate term by term. But, that seems too complicated, so look for other alternatives.

Notice that $(1 + x^4)^5$ appears in both parts of the problem so the term $(1 + x^4)$ is an *"inside"* function. Try $u = 1 + x^4$, then $du = 4x^3\ dx$. Since the term $(x^3\ dx)$ appears in part (a), that integral will be easy to solve. It becomes

$$\int x^3(1+x^4)^5 dx = \frac{1}{4}\int (1+x^4)^5(4x^3 dx) = \frac{1}{4}\int u^5 du$$
$$= (\frac{1}{4})(\frac{1}{6}u^6) + C = \frac{1}{24}(1+x^4)^6 + C.$$

* * * WARNING * * *

The sample problems you've worked through in this chapter may have seemed very easy, because you were on guard for simple solutions. The moral of this chapter is:

When you start working on a problem, always check for an easy algebraic manipulation or obvious substitution. Only when you're sure the problem cannot be SIMPLIFIED should you try anything else.

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EXERCISES FOR CHAPTER 1

In each of the following exercises, one problem can be done easily. Use the techniques of easy algebraic manipulations and obvious substitutions to determine which it is, and solve it.

1. (a) $\int \frac{x+1}{x^3+x^2+1}dx$ (b) $\int \frac{x^3+x^2+1}{x+1}dx$

2. (a) $\int \ln(e^x)dx$ (b) $\int \ln(x)dx$

3. (a) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^5}dx$ (b) $\int \frac{1}{(1+\sqrt{x})^5}dx$

4. (a) $\int \frac{e^x}{e^{5x}-1}dx$ (b) $\int \frac{e^{5x}-1}{e^x}dx$

5. (a) $\int \frac{1}{x^2-4x+3}dx$ (b) $\int \frac{x-2}{x^2-4x+3}dx$

CHAPTER 2

CLASSIFY !

As noted in the introduction, experts generally follow a 3-step procedure when solving integrals. The first step consists of looking for simplifications or easy solutions to a problem. The second step, if necessary, consists of choosing and applying the technique most likely to solve a problem.

This choice of technique is usually based on the FORM of the integrand. The solution to a problem follows routinely once the right technique has been chosen.

In this chapter we will CLASSIFY integrals into 2 basic categories, and discuss the techniques most often effective in dealing with them. Our classification is summarized by the second box in the General Procedure:

Step 2:	**CLASSIFY**
Rational Functions	Products

Your goal in working through this section should be to classify integrands by form and recall the techniques appropriate to them. If you systematically use the simplifications of Chapter 1 and the classification scheme of this section, you should be able to solve most of the problems at the end of your text's chapter on integration.

RATIONAL FUNCTIONS

A rational function is the quotient of two polynomials. The procedure for integrating rational functions is straightforward, although it may sometimes be long and involved. A large part of that procedure is purely algebraic, and consists of *"breaking up"* complicated rational functions into sums of simpler ones. We will begin by examining the simple or *"basic"* rational functions, and then discuss how to break up the more complicated ones.

BASIC RATIONAL FUNCTIONS

A Basic Rational Function is a *"proper fraction"* of the form

$$\frac{r}{ax+b}\;;\quad \frac{r}{(ax+b)^n}\;,\quad \frac{rx+s}{ax^2+bx+c}.$$

Basic rational functions of the first two types are easy to integrate. If the denominator is (ax + b) or $(ax + b)^n$, the substitution u = (ax + b) will solve the problem.

SAMPLE PROBLEMS

The solutions to these problems illustrate the technique described above. Try to solve them before you read the solutions. If they cause you a great deal of difficulty, you should probably practice on some similar problems from your textbook.

1. $\int \frac{4}{5x+7}dx$

2. $\int \frac{5}{(4x+3)^6}dx$

3. $\int \frac{x+2}{x^2+4x+13}dx$

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SOLUTIONS

1. $\int \frac{4}{5x+7}dx$

There is no algebraic simplification possible. Since the denominator is (5x + 7), we make the substitutions

$$u = 5x + 7; \quad du = 5\,dx$$

The integral then becomes

$$\frac{4}{5}\int \frac{5\,dx}{5x+7} = \frac{4}{5}\int \frac{du}{u} = \frac{4}{5}\ln u + C$$
$$= \frac{4}{5}\ln 5x+7 + C.$$

2. $\int \frac{5}{(4x+3)^6}dx$

Again, there is no algebraic simplification. Since the denominator is $(4x + 3)^6$, the substitutions

$$u = 4x + 3; \quad du = 4\,dx$$

are called for. The integral then becomes:

$$\frac{5}{4}\int \frac{4dx}{(4x+3)^6} = \frac{5}{4}\int \frac{du}{u^6} = \frac{5}{4}\int u^{-6}du = \frac{5}{4}\left(\frac{u^{-5}}{-5}\right) + C$$
$$= -\frac{1}{4}\left(\frac{1}{u^5}\right) + C = -\frac{1}{4(4x+3)^5} + C.$$

3. $\int \frac{x+2}{x^2+4x+13}dx$

As always, begin work on this problem by looking for easy algebraic manipulations. The integral can be broken into a sum of two integrals, but this does not look especially promising. This is already a "*proper fraction*," so look for obvious substitutions next.

The "*nasty*" term is the denominator, so consider the substitution

$$u = x^2 + 4x + 13.$$

This would give

$$du = (2x + 4)\,dx,$$

which is double the numerator in this problem! The rest is easy. The integral is

$$\frac{1}{2}\int\frac{(2x+4)\,dx}{x^2+4x+13} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln u + C = \frac{1}{2}\ln\ x^2+4x+13\ + C.$$

PRODUCTS

If the integrand is a product, and especially if the integrand is a product of dissimilar functions, you should consider using integration by parts to solve the problem. The formula is derived from the formula for the differential of a product,

$$d(uv) = u\,dv + v\,du$$

Integrating each term, we obtain

$$uv = u\,dv + \int v\,du.$$

Rearranging this gives

$$u\,dv = uv - \int v\,du.$$

To apply this formula, we separate the integrand into two parts. We call one u and the other dv. We differentiate u to obtain du, and integrate dv to obtain v. If we can then integrate the term $\int v\,du$, the problem is solved. The goal of this procedure, then, is to choose u and dv such that the term v du is easier to solve than the original problem. As the sample problems illustrate, this usually happens when u is simplified by differentiation. These

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comments are summarized in the following chart.

INTEGRATING PRODUCTS

Consider integration by parts. The formula is

$$u\,dv = uv - \int v\,du.$$

and your choice of u and dv should be governed by two things:

(1) You must be able to integrate the term you call dv.

(2) You want $\int v\,du$ to be easier than the original integral. This often happens when u is simplified by differentiation.

NOTE: This formula also has special application to the integration of single terms that we can't integrate otherwise. Since $\int f(x)dx$ can be written as $\int [f(x)][1\,dx]$, we can think of that integrand as a product and try integration by parts with $u = f(x)$ and $dv = dx$.

SAMPLE PROBLEMS

As usual, try these problems before you read the solutions. Pay particular attention to the reasoning used in making the choices of u and dv in each problem.

1. $\int x e^{3x} dx$

2. $\int (\ln x)^2 dx$

SOLUTIONS

1. $\int x e^{3x} dx$

There are two possible choices of u and dv in this problem.

$$\left| \begin{array}{l} u = x \\ \\ dv = e^{3x}\ dx \end{array} \right| \quad \textbf{OR} \quad \left| \begin{array}{l} u = e^{3x} \\ \\ dv = x\ dx \end{array} \right|$$

Determining du and v in each we get

$$\left| \begin{array}{l} du = dx \\ \\ v = 1/3\ e^{3x} \end{array} \right| \quad \text{for the first, and}$$

$$\left| \begin{array}{l} du = 3e^{3x}\ dx \\ \\ v = 1/2\ x^2 \end{array} \right| \quad \text{for the second.}$$

Clearly, $\int v\ du$ is easier to solve in the first case, so we make the substitutions $u = x$, $dv = e^{3x}\ dx$. Then

$$\int (x)(e^{3x}dx) = (x)(\frac{1}{3}e^{3x}) - \int (\frac{1}{3}e^{3x})dx = \frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x} + C.$$

2. $\int (\ln x)^2 dx$

This problem can be done by parts if we write it as

$$\int [(\ln x)^2][1dx]$$

With

$u = (\ln x)^2$	AND	$du = 2/x \ln x \, dx$
$dv = 1\, dx$		$v = x$

we obtain

$$\int (\ln x)^2(1dx) = (\ln x)^2(x) - \int (x)(\frac{2}{x}\ln x\, dx) = x(\ln x)^2 - 2\int \ln x\, dx.$$

We haven't solved the problem, but we've simplified it. We now have to integrate $\int \ln x \, dx$ instead of $\int (\ln x)^2 \, dx$.

A second integration by parts with $U = \ln x$, $dV = 1\, dx$ gives

$$x(\ln x)^2 - 2\int (\ln x)(1dx) = x(\ln x)^2 - 2[(\ln x)(x) - \int (x)(\frac{1}{x}dx)]$$

$$= \quad 2(\ln x)^2 - 2x(\ln x) + 2x + C.$$

NOTE: Like many problems in integration, this can be done in more than one way. The substitution $w = \ln x$ (or $e^w = x$) transforms $\int (\ln x)^2 \, dx$ to $\int w^2 e^w \, dw$, which is done by parts (twice).

CHAPTER 3

MODIFY !

Chapters 1 and 2 of this booklet contain the basic techniques necessary for solving most business calculus integration problems. Once we can **SIMPLIFY** or **CLASSIFY** an integrand, its solution is a routine (although not necessarily easy) matter.

We encounter the most difficulty with problems of unfamiliar form, those which resist classification by the methods of Chapter 2. With such problems our goal is to **MODIFY** the integrand, manipulating it until it is in a more convenient or recognizable form. Once this has been done, we return to the **SIMPLIFY** and **CLASSIFY** steps of the General Procedure to finish the problem.

Problems of this type are generally not encountered in a typical Business Calculus course. Only the basic ideas are presented in this chapter, with additional information, examples, and problems available in the original module.

The three sections of this chapter are:

(1) Problem Similarities: looking for and exploiting resemblances between the problem we are working on and problems we know how to integrate
(2) Special Manipulations: techniques for expressing complicated integrands in more convenient form
(3) Needs Analysis: looking to see what additional terms might help solve a problem, and modifying the integrand to include them.

Together, these form the third step of the General Procedure:

Step 3:	**MODIFY**	
Problem Similarities	Special Manipulations	Needs Analysis

PROBLEM SIMILARITIES

Some integrals can be classified easily, but look so complicated that the standard procedures for solving them promise to be very messy. Other integrals may not fit into the classification scheme of Chapter 2, and we may not know an appropriate way to solve them. One way to approach such problems is to look for similarities between them and problems we know how to do. If the form of a difficult problem resembles that of a *"standard"* problem, there are two possibilities. We might be able to reduce the difficult problem to that *"standard"* form. Or, the techniques we would use on the easier problem might help us solve the more difficult one.

Warning: There are integration problems that appear very similar to a manageable one, but on closer scrutiny, are impossible to solve using the techniques presented in this module. Additional information will be necessary to solve these types of problems.

PROBLEM SIMILARITIES
(1) Look for easy problems similar to the one you are working on. (2) Try to reduce the difficult problem to the form of the easy similar problem. (3) Try the techniques you would use on the similar problem.

SPECIAL MANIPULATIONS

In this section we discuss four techniques designed to express complicated integrands in more convenient form for integration. They are:

SPECIAL MANIPULATIONS
1. Rat nalizing denominators of quotients 2. Spe use of trigonometric identities 3. *"Common denominator"* substitutions 4. *"Desperation"* substitutions

These techniques often involve complex manipulations. It may not be clear that they are helping to solve a problem until we have done some complicated calculations. For that reason these techniques differ from the simplifications of Chapter 1.

When we first examine an integral, we look for fast and easy ways to solve it. If that fails, we try to classify it and use standard techniques. Only if that fails, or if the standard techniques look very complicated, do we look for alternatives such as these. With practice you will discover which approaches to integrals you can examine rapidly, and which are time-consuming. This knowledge should govern the order in which you apply them.

NEEDS ANALYSIS

The technique of needs analysis has been implicit in much of our work so far, and we now state it formally as an integration technique. It consists of asking what might enable us to solve a problem, and then either adding it (and compensating for it) or changing something in the problem to it. For an integrand involving e^x, we might seek to introduce (e^x dx). [This is done automatically by the substitutions $u = e^x$; $du = e^x dx$; $dx = 1/u\ du$.] If the integrand involves x^n, we can look for a way to introduce [nx^{n-1} dx]. As usual, we summarize in table form.

NEEDS ANALYSIS
(1) Look for a term, or a form of the integral, that would enable you to solve it. (2) Try to modify the integral to produce the term or form you need. (3) Try to introduce the term you need, and compensate for it.

APPENDIX G

STATISTICAL ANALYSES

EXPERIMENTAL VS. CONTROL, TREATMENT AND GENDER - POSTTEST I

General Linear Models Procedure

Dependent Variable: TEST1

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	1090.69874500	363.56624833	5.15	0.0001
Error	91	3601.13283394	39.57288829		
Corrected Total	94	4691.83157895			

R-Square	C.V.	Root MSE	TEST1 Mean
0.232468	50.90429	6.29069855	12.35789474

Source	DF	Type I SS	Mean Square	F Value	Pr > F
GENDER	1	517.16491228	517.16491228	13.07	0.0005
TIME	1	0.30143301	0.30143301	0.01	0.9305
TRTMENT	1	573.23239971	573.23239971	14.45	0.0003

Source	DF	Type III SS	Mean Square	F Value	Pr > F
GENDER	1	533.69435157	533.69435157	13.49	0.0004
TIME	1	5.03909403	5.03909403	0.13	0.7220
TRTMENT	1	573.23239971	573.23239971	14.45	0.0003

EXPERIMENTAL VS. CONTROL, TREATMENT AND GENDER - POSTTEST I

General Linear Models Procedure
Least Squares Means

TRTMENT	TEST1 LSMEAN
1	9.4197818
2	14.3676054

BEST COPY AVAILABLE

INTERACTIONS - POSTTEST I

General Linear Models Procedure

Dependent Variable: TEST1

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	1403.85065203	200.55009315	5.31	0.0001
Error	87	3287.98092692	37.79288422		
Corrected Total	94	4691.83157895			

	R-Square	C.V.	Root MSE	TEST1 Mean
	0.299212	49.74627	6.14759174	12.35789474

Source	DF	Type I SS	Mean Square	F Value	Pr > F
GENDER	1	517.16491228	517.16491228	13.68	0.0004
TIME	1	0.30143301	0.30143301	0.01	0.9290
GENDER*TIME	1	50.67341437	50.67341437	1.34	0.2501
TRTMENT	1	530.91408580	530.91408580	14.05	0.0003
GENDER*TRTMENT	1	108.96486384	108.96486384	2.88	0.0931
TIME*TRTMENT	1	0.04547686	0.04547686	0.00	0.9724
GENDER*TIME*TRTMENT	1	195.78646088	195.78646088	5.18	0.0253

Source	DF	Type III SS	Mean Square	F Value	Pr > F
GENDER	1	27.39920621	27.39920621	0.72	0.3969
TIME	1	45.84235633	45.84235633	1.21	0.2738
GENDER*TIME	1	88.99560385	88.99560385	2.35	0.1285
TRTMENT	1	70.35099516	70.35099516	1.86	0.1760
GENDER*TRTMENT	1	302.03108914	302.03108914	7.99	0.0058
TIME*TRTMENT	1	32.15276698	32.15276698	0.85	0.3589
GENDER*TIME*TRTMENT	1	195.78646088	195.78646088	5.18	0.0253

INTERACTIONS BETWEEN GENDER AND TREATMENT

General Linear Models Procedure

Dependent Variable: TEST1

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	1192.73334385	397.57778128	10.34	0.0001
Error	91	3499.09823510	38.45162395		
Corrected Total	94	4691.83157895			

R-Square	C.V.	Root MSE	TEST1 Mean
0.254215	50.17795	6.20093775	12.35789474

Source	DF	Type I SS	Mean Square	F Value	Pr > F
GENDER*TRTMENT	3	1192.73334385	397.57778128	10.34	0.0001

Source	DF	Type III SS	Mean Square	F Value	Pr > F
GENDER*TRTMENT	3	1192.73334385	397.57778128	10.34	0.0001

INSTRUCTOR DIFFERENCES - POSTTEST I

--- TRTMENT=1 ---

General Linear Models Procedure

Dependent Variable: TEST1

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	387.30587855	77.46117571	3.15	0.0181
Error	37	909.76388889	24.58821321		
Corrected Total	42	1297.06976744			

R-Square	C.V.	Root MSE	TEST1 Mean
0.298601	51.13237	4.95865034	9.69767442

Source	DF	Type I SS	Mean Square	F Value	Pr > F
INSTRCTR	2	103.89417220	51.94708610	2.11	0.1353
GENDER	1	52.04278895	52.04278895	2.12	0.1541
INSTRCTR*GENDER	2	231.36891740	115.68445870	4.70	0.0151

Source	DF	Type III SS	Mean Square	F Value	Pr > F
INSTRCTR	2	151.15650689	75.57825345	3.07	0.0582
GENDER	1	11.05586055	11.05586055	0.45	0.5067
INSTRCTR*GENDER	2	231.36891740	115.68445870	4.70	0.0151

INSTRUCTOR DIFFERENCES - POSTTEST I

-- TRTMENT=2 --

General Linear Models Procedure

Dependent Variable: TEST1

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	758.86477411	151.77295482	3.36	0.0115
Error	46	2079.96214896	45.21656846		
Corrected Total	51	2838.82692308			

R-Square	C.V.	Root MSE	TEST1 Mean
0.267316	46.19088	6.72432662	14.55769231

Source	DF	Type I SS	Mean Square	F Value	Pr > F
INSTRCTR	2	135.04326923	67.52163462	1.49	0.2353
GENDER	1	604.08988559	604.08988559	13.36	0.0007
INSTRCTR*GENDER	2	19.73161929	9.86580965	0.22	0.8048

Source	DF	Type III SS	Mean Square	F Value	Pr > F
INSTRCTR	2	164.79491043	82.39745522	1.82	0.1731
GENDER	1	594.52055925	594.52055925	13.15	0.0007
INSTRCTR*GENDER	2	19.73161929	9.86580965	0.22	0.8048

EXPERIMENTAL VS. CONTROL, TREATMENT AND GENDER - POSTTEST II

General Linear Models Procedure

Dependent Variable: TEST2

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	708.61194008	236.20398003	6.38	0.0006
Error	91	3369.11437571	37.02323490		
Corrected Total	94	4077.72631579			

R-Square	C.V.	Root MSE	TEST2 Mean
0.173776	60.15024	6.08467213	10.11578947

Source	DF	Type I SS	Mean Square	F Value	Pr > F
GENDER	1	120.26842572	120.26842572	3.25	0.0749
TIME	1	14.58839747	14.58839747	0.39	0.5318
TRTMENT	1	573.75511689	573.75511689	15.50	0.0002

Source	DF	Type III SS	Mean Square	F Value	Pr > F
GENDER	1	132.54702140	132.54702140	3.58	0.0617
TIME	1	30.33571093	30.33571093	0.82	0.3678
TRTMENT	1	573.75511689	573.75511689	15.50	0.0002

EXPERIMENTAL VS. CONTROL, TREATMENT AND GENDER - POSTTEST II

General Linear Models Procedure
Least Squares Means

TRTMENT	TEST2 LSMEAN
1	6.8912193
2	11.84[illegible]983

INTERACTIONS - POSTTEST II

General Linear Models Procedure

Dependent Variable: TEST2

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	985.07738531	140.72534076	3.96	0.0008
Error	87	3092.64893046	35.54768886		
Corrected Total	94	4077.72631579			

R-Square	C.V.	Root MSE	TEST2 Mean
0.241575	58.93943	5.96218826	10.11578947

Source	DF	Type I SS	Mean Square	F Value	Pr > F
GENDER	1	120.26842572	120.26842572	3.38	0.0693
TIME	1	14.58839747	14.58839747	0.41	0.5235
GENDER*TIME	1	46.80145231	46.80145231	1.32	0.2544
TRTMENT	1	533.74824964	533.74824964	15.01	0.0002
GENDER*TRTMENT	1	141.82349778	141.82349778	3.99	0.0489
TIME*TRTMENT	1	37.19797170	37.19797170	1.05	0.3092
GENDER*TIME*TRTMENT	1	90.64939070	90.64939070	2.55	0.1139

Source	DF	Type III SS	Mean Square	F Value	Pr > F
GENDER	1	5.25474531	5.25474531	0.15	0.7016
TIME	1	0.20212918	0.20212918	0.01	0.9401
GENDER*TIME	1	27.63132991	27.63132991	0.78	0.3804
TRTMENT	1	207.79637322	207.79637322	5.85	0.0177
GENDER*TRTMENT	1	201.43633211	201.43633211	5.57	0.0195
TIME*TRTMENT	1	3.56496018	3.56496018	0.10	0.7522
GENDER*TIME*TRTMENT	1	90.64939070	90.64939070	2.55	0.1139

INTERACTIONS BETWEEN GENDER AND TREATMENT

General Linear Models Procedure

Dependent Variable: TEST2

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	831.42311899	277.14103966	7.77	0.0001
Error	91	3246.30319680	35.67366150		
Corrected Total	94	4077.72631579			

	R-Square	C.V.	Root MSE	TEST2 Mean
	0.203894	59.04377	5.97274321	10.11578947

Source	DF	Type I SS	Mean Square	F Value	Pr > F
GENDER*TRTMENT	3	831.42311899	277.14103966	7.77	0.0001

Source	DF	Type III SS	Mean Square	F Value	Pr > F
GENDER*TRTMENT	3	831.42311899	277.14103966	7.77	0.0001

INSTRUCTOR DIFFERENCES - POSTTEST II

-- TRTMENT=1 --

General Linear Models Procedure

Dependent Variable: TEST2

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	216.80878553	43.36175711	1.02	0.4218
Error	37	1577.88888889	42.64564565		
Corrected Total	42	1794.69767442			

	R-Square	C.V.	Root MSE	TEST2 Mean
	0.120805	87.47839	6.53036336	7.46511628

Source	DF	Type I SS	Mean Square	F Value	Pr > F
INSTRCTR	2	109.24529347	54.62264673	1.28	0.2898
GENDER	1	0.00051142	0.00051142	0.00	0.9973
INSTRCTR*GENDER	2	107.56298064	53.78149032	1.26	0.2952

Source	DF	Type III SS	Mean Square	F Value	Pr > F
INSTRCTR	2	50.71731636	25.35865818	0.59	0.5569
GENDER	1	28.85885886	28.85885886	0.68	0.4161
INSTRCTR*GENDER	2	107.56298064	53.78149032	1.26	0.2952

INSTRUCTOR DIFFERENCES - POSTTEST II

-- TRTMENT=2 --

General Linear Models Procedure

Dependent Variable: TEST2

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	297.71978022	59.54395604	1.91	0.1108
Error	46	1433.35714286	31.15993789		
Corrected Total	51	1731.07692308			

R-Square	C.V.	Root MSE	TEST2 Mean
0.171985	45.35463	5.58210873	12.30769231

Source	DF	Type I SS	Mean Square	F Value	Pr > F
INSTRCTR	2	0.38557692	0.19278846	0.01	0.9938
GENDER	1	281.48102483	281.48102483	9.03	0.0043
INSTRCTR*GENDER	2	15.85317847	7.92658923	0.25	0.7765

Source	DF	Type III SS	Mean Square	F Value	Pr > F
INSTRCTR	2	6.87946863	3.43973432	0.11	0.8957
GENDER	1	283.80170145	283.80170145	9.11	0.0041
INSTRCTR*GENDER	2	15.85317847	7.92658923	0.25	0.7765

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DIFFERENCES BETWEEN POSTTEST I AND POSTTEST II

General Linear Models Procedure

Dependent Variable: DIFF

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	211.41669481	30.20238497	0.75	0.6288
Error	87	3494.01488414	40.16109062		
Corrected Total	94	3705.43157895			

R-Square	C.V.	Root MSE	DIFF Mean
0.057056	282.6485	6.33727786	2.24210526

Source	DF	Type I SS	Mean Square	F Value	Pr > F
GENDER	1	138.64035554	138.64035554	3.45	0.0666
TIME	1	10.69582306	10.69582306	0.27	0.6071
GENDER*TIME	1	0.07693270	0.07693270	0.00	0.9652
TRTMENT	1	0.00377232	0.00377232	0.00	0.9523
GENDER*TRTMENT	1	2.16190919	2.16190919	0.05	0.8171
TIME*TRTMENT	1	39.84471507	39.84471507	0.99	0.3220
GENDER*TIME*TRTMENT	1	19.99318693	19.99318693	0.50	0.4823

Source	DF	Type III SS	Mean Square	F Value	Pr > F
GENDER	1	8.65596409	8.65596409	0.22	0.6436
TIME	1	39.95643885	39.95643885	0.99	0.3213
GENDER*TIME	1	17.44897471	17.44897471	0.43	0.5115
TRTMENT	1	36.33188793	36.33188793	0.90	0.3442
GENDER*TRTMENT	1	10.15196526	10.15196526	0.25	0.6164
TIME*TRTMENT	1	57.13018463	57.13018463	1.42	0.2362
GENDER*TIME*TRTMENT	1	19.99318693	19.99318693	0.50	0.4823

DIFFERENCES BETWEEN POSTTEST I AND POSTTEST II

General Linear Models Procedure
Least Squares Means

TRTMENT	DIFF LSMEAN
1	4.15882353
2	2.12537879

INSTRUCTOR DIFFERENCES IN POSTTEST I VS. POSTTEST II

TRTMENT=1

General Linear Models Procedure

Dependent Variable: DIFF

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	163.63275194	32.72655039	0.90	0.4916
Error	37	1346.04166667	36.37950450		
Corrected Total	42	1509.67441860			

R-Square	C.V.	Root MSE	DIFF Mean
0.108389	270.1628	6.03154246	2.23255814

Source	DF	Type I SS	Mean Square	F Value	Pr > F
INSTRCTR	2	84.92739480	42.46369740	1.17	0.3224
GENDER	1	52.36958745	52.36958745	1.44	0.2378
INSTRCTR*GENDER	2	26.33576969	13.16788484	0.36	0.6987

Source	DF	Type III SS	Mean Square	F Value	Pr > F
INSTRCTR	2	74.45868138	37.22934069	1.02	0.3693
GENDER	1	4.19024493	4.19024493	0.12	0.7362
INSTRCTR*GENDER	2	26.33576969	13.16788484	0.36	0.6987

INSTRUCTOR DIFFERENCES IN POSTTEST I VS. POSTTEST II

-- TRTMENT=2 --

General Linear Models Procedure

Dependent Variable: DIFF

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	5	186.05708181	37.21141636	0.85	0.5206
Error	46	2009.69291819	43.68897648		
Corrected Total	51	2195.75000000			

	R-Square	C.V.	Root MSE	DIFF Mean
	0.084735	293.7673	6.60976372	2.25000000

Source	DF	Type I SS	Mean Square	F Value	Pr > F
INSTRCTR	2	121.11153846	60.55576923	1.39	0.2603
GENDER	1	60.85316456	60.85316456	1.39	0.2440
INSTRCTR*GENDER	2	4.09237879	2.04618939	0.05	0.9543

Source	DF	Type III SS	Mean Square	F Value	Pr > F
INSTRCTR	2	130.45070401	65.22535200	1.49	0.2354
GENDER	1	56.79698108	56.79698108	1.30	0.2601
INSTRCTR*GENDER	2	4.09237879	2.04618939	0.05	0.9543

MEAN TEST SCORES BY GENDER AND POSTTEST

TRTMENT	GENDER	N Obs	Variable	N	Mean	Std Dev	Minimum	Maximum
1	1	21	TEST1	21	8.47619	4.83342	0.00000	20.00000
			TEST2	21	7.71429	5.91729	0.00000	19.00000
	2	22	TEST1	22	10.86364	6.04976	2.00000	29.00000
			TEST2	22	7.22727	7.21065	0.00000	33.00000
2	1	26	TEST1	26	11.23077	6.56236	2.00000	26.00000
			TEST2	26	10.00000	4.56946	2.00000	19.00000
	2	26	TEST1	26	17.88462	6.88957	8.00000	36.00000
			TEST2	26	14.61538	6.10624	2.00000	30.00000

GENDER

1 = male
2 = female

TRTMENT

1 = control
2 = experimental