P2 - FVV - Prof. Edson Iwaki - 23/04/2015

Nome:

1. (**2.0**) Esboce a região de integração e inverta a ordem de integração da integral $\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x,y) \quad dydx.$

2. (**2.0**) Encontre o máximo e mínimo global da função $f(x,y) = e^{x^2+y^2+y}$, $|x| \leq 1, |y| \leq 1$.

3. (**3.0**) Calcule $\int\int_R (8-x-y) \quad dxdy$, onde R é a região delimitada por $y = x^2$ e $y = 4$.

4. (**3.0**) Calcule $\int\int_B x \quad dxdy$, onde $B = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2-x \leq 0\}$.

5. (**1.5**) Calcule $\int\int_B y \quad dxdy$, onde B é a região compreendida entre os gráficos de $y = x$ e $y = x^2$, com $0 \leq x \leq 2$.

6. (EXTRA) (**3.0**) Calcule $\int\int_B \sqrt[3]{y^2-x^2} \quad dxdy$, onde B é o paralelogramo de vértices $(0,0), (\frac{1}{2}, \frac{1}{2}), (0,1), (-\frac{1}{2}, \frac{1}{2})$.

<u>GABARITO</u>:

4).
$$\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x,y)\,dy\,dx.$$

- Determinação da região de Integração: B:

Para cada x fixado, $x \in [0,1]$,
y varia de $\sqrt{x-x^2}$ a $\sqrt{2x}$.

$\therefore$ $B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, \sqrt{x-x^2} \le y \le \sqrt{2x}\}$.

• $y = \sqrt{x-x^2}$

$y^2 = x - x^2 \iff x^2 - x + y^2 = 0 \iff x^2 - x + \frac{1}{4} - \frac{1}{4} + y^2 = 0$

$\Updownarrow$

$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$

Circunferência de centro $C(1/2, 0)$ e raio $r = 1/2$.

•• $y = \sqrt{2x}$

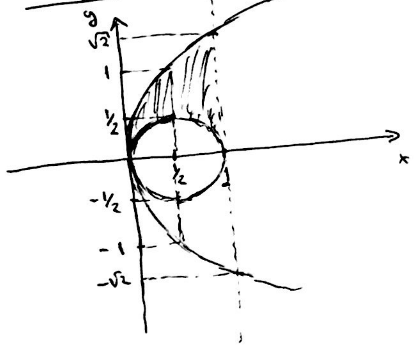
$y^2 = 2x$

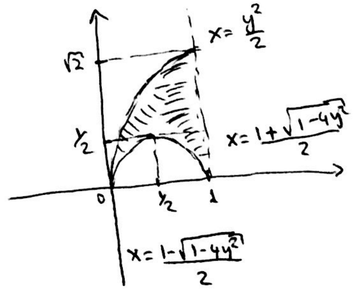
••• Expressando x em função de y, obtemos.

$x^2 - x + y^2 = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot y^2}}{2} \Rightarrow x = \dfrac{1 \pm \sqrt{1-4y^2}}{2}$

$y^2 = 2x \Rightarrow x = \dfrac{y^2}{2}$.

• <u>Esboço de B</u>:





$$\therefore \int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x,y)\,dy\,dx = \iint_{B_1} f(x,y)\,dx\,dy + \iint_{B_2} f(x,y)\,dx\,dy + \iint_{B_3} f(x,y)\,dx\,dy$$

, onde $B_1 = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{1}{2},\ \frac{y^2}{2} \leq x \leq \frac{1-\sqrt{1-4y^2}}{2}\}$

$B_2 = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{1}{2},\ \frac{1+\sqrt{1-4y^2}}{2} \leq x \leq 1\}$

$B_3 = \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{2} \leq y \leq \sqrt{2},\ \frac{y^2}{2} \leq x \leq 1\}$.

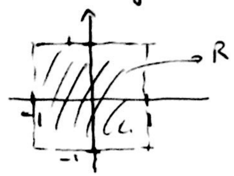
$$\therefore \int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x,y)\,dy\,dx =$$

$$= \int_0^{1/2} \left[\int_{y^2/2}^{\frac{1-\sqrt{1-4y^2}}{2}} f(x,y)\,dx \right] dy + \int_0^{1/2} \left[\int_{\frac{1+\sqrt{1-4y^2}}{2}}^{1} f(x,y)\,dx \right] dy +$$

$$+ \int_{1/2}^{\sqrt{2}} \left[\int_{y^2/2}^{1} f(x,y)\,dx \right] dy. \ \square$$

02) Encontre o máximo e mínimo global da função $f(x,y) = e^{x^2+y^2+y}$, $|x| \le 1$, $|y| \le 1$.

Seja R a região def por $\{(x,y) \mid |x| \le 1, |y| \le 1\}$.



- Pontos críticos no interior de R:

$f_x = (e^{x^2+y^2+y}) \cdot 2x$

$f_y = (e^{x^2+y^2+y}) \cdot (2y+1)$

$f_{xx} = (e^{x^2+y^2+y})\, 2x \cdot 2x + (e^{x^2+y^2+y}) \cdot 2 = (e^{x^2+y^2+y})(4x^2+2)$

$f_{xy} = (e^{x^2+y^2+2})(2y+1)(2x)$

$f_{yy} = (e^{x^2+y^2+2})(2y+1)(2y+1) + (e^{x^2+y^2+2}) \cdot 2 =$

$= (e^{x^2+y^2+y})((2y+1)^2+2)$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow x = 0$ e $y = -\frac{1}{2}$

$\therefore (0, -\frac{1}{2})$ é o único ponto crítico no interior de R.

$H(0, -\frac{1}{2}) = \det \begin{bmatrix} 2e^{-1/4} & 0 \\ 0 & 2e^{-1/4} \end{bmatrix} = 4e^{-1/2} > 0$

$\therefore (0, -\frac{1}{2})$ ponto de mínimo local de f

e $f(0, -\frac{1}{2}) = e^{-1/4}$.

Scanned by CamScanner

- Pontos críticos na fronteira de R :

Lado 1: $\{(1,y) \mid -1 \leq y \leq 1\}$.

Def. $g_1(y) = f(1,y) = e^{1+y^2+y}$

$$g_1'(y) = e^{1+y^2+y} \cdot (2y+1)$$

$$g_1'(y) = 0 \Rightarrow y = -\frac{1}{2}$$

e $g_1\left(-\frac{1}{2}\right) = e^{1+\frac{1}{4}-\frac{1}{2}} = e^{3/4}$.

Temos de verificar os valores de g_1 nos extremos do intervalo.

$$g_1(-1) = e^{1+1-1} = e$$

$$g_1(1) = e^{1+1+1} = e^3$$

Lado 2 : $\{(-1,y) \mid -1 \leq y \leq 1\}$.

Def $g_2(y) = f(-1,y) = e^{1+y^2+y}$.

De maneira análoga ao caso anterior,

$$g_2'(y) = 0 \Leftrightarrow y = -\frac{1}{2}$$

$$g_2\left(-\frac{1}{2}\right) = e^{3/4}$$

$$g_2(-1) = e$$

$$g_2(1) = e^3$$

Lado 3: $\{(x,1) \mid -1 \leq x \leq 1\}$.

Def $g_3(x) = f(x,1) = e^{x^2+2}$

$$g_3'(x) = e^{x^2+2} \cdot 2x$$

$$g_3'(x) = 0 \Rightarrow x = 0$$

$g_3(0) = 0$,

$$g_3(-1) = e^3$$

$g_3(1) = e^3$.

Lado 4: $\{(x,-1) \mid -1 \leq x \leq 1\}$.

Def. $g_4(x) = f(x,-1) = e^{x^2}$

$g_4'(x) = e^{x^2} \cdot 2x$

$g_4'(x) = 0 \Rightarrow x = 0$

$g_4(0) = 1.$

$g_4(-1) = e$

$g_4(1) = e$

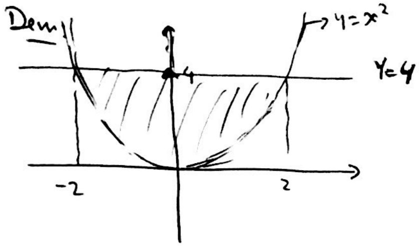
Da análise dos casos feita acima, podemos concluir que:

f admite valor máximo e^3 nos pontos $(1,1)$, $(-1,1)$

e f admite valor mínimo $e^{-1/4}$ no ponto $(0,-1/2)$.

03) $\iint_R (8-x-y)\,dx\,dy$, R é a região delimitada por $y=x^2$ e $y=4$

Dem:



$$\iint_R (8-x-y)\,dx\,dy = \int_{-2}^{2}\left[\int_{x^2}^{4}(8-x-y)\,dy\right]dx =$$

$$= \ldots = \frac{896}{15}.$$

04) $\iint_B x \, dx\,dy$, $\quad B = \{(x,y) \mid x^2+y^2-x \le 0\}$.

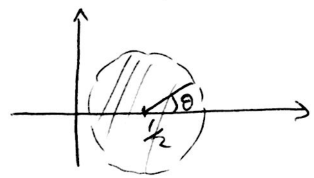
$$x^2+y^2-x=0 \Leftrightarrow x^2-x+\frac{1}{4}-\frac{1}{4}+y^2=0$$
$$\Updownarrow$$
$$\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}.$$

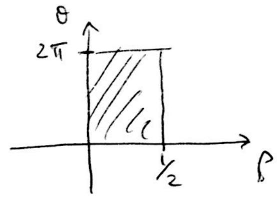
$\therefore B=\{(x,y)\in\mathbb{R}^2 \mid \left(x-\frac{1}{2}\right)^2+y^2 \le \frac{1}{4}\}$.

- Mudança de variável.

$$\begin{cases} x-\frac{1}{2}=\rho\cos\theta \\ y=\rho\,\mathrm{sen}\,\theta \end{cases}$$

$dx\,dy=\rho\,d\rho\,d\theta$, $\quad B_{\rho\theta}=\{(\rho,\theta) \mid 0\le\rho\le\frac{1}{2},\ 0\le\theta\le 2\pi\}$.





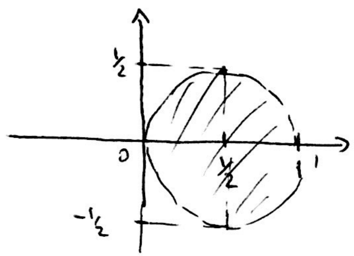
$$\therefore \iint_B x\,dx\,dy = \iint_{B_{\rho\theta}} \left(\frac{1}{2}+\rho\cos\theta\right)\rho\,d\rho\,d\theta =$$

$$=\int_0^{2\pi}\left[\int_0^{1/2}\left(\frac{\rho}{2}+\rho^2\cos\theta\right)d\rho\right]d\theta = \int_0^{2\pi}\left(\frac{1}{16}+\frac{1}{24}\cos\theta\right)d\theta=\frac{\pi}{8}$$

Q4) <u>Solução 2</u>:

$$x^2+y^2-x=0 \iff \left(x-\tfrac{1}{2}\right)^2+y^2=\tfrac{1}{4}.$$

$$B=\left\{(x,y) \mid \left(x-\tfrac{1}{2}\right)^2+y^2 \le \tfrac{1}{4}\right\}.$$



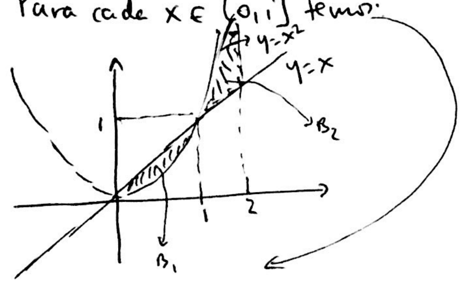
$$x^2+y^2-x=0 \iff y^2=x-x^2$$
$$\Updownarrow$$
$$y=\pm\sqrt{x-x^2}$$

$$\therefore \iint_B x\,dx\,dy = \int_0^1 \left[\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} x\,dy\right]dx = \ldots$$

$$= \int_0^1 2x\sqrt{x-x^2}\,dx = \ldots = \frac{\pi}{8}.$$

05) $\iint_B y \, dx\,dy$, B é a região compreendida entre os gráficos de $y = x$ e $y = x^2$, com $0 \le x \le 2$.

a) Para cada $x \in [0,1]$ temos:



$$\iint_{B_1} y \, dy\,dx = \int_0^1 \left[\int_{x^2}^{x} y \, dy \right] dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx = \cdots = \frac{1}{15}.$$

b) Para cada $x \in [1,2]$

$$\iint_{B_2} y \, dy\,dx = \int_1^2 \left[\int_x^{x^2} y \, dy \right] dx = \int_1^2 \left(\frac{x^4}{2} - \frac{x^2}{2} \right) dx = \cdots = \frac{29}{15}$$

$$\therefore \iint_B y \, dy\,dx = \iint_{B_1} y \, dy\,dx + \iint_{B_2} y \, dy\,dx = \frac{1}{15} + \frac{29}{15} = 2.$$

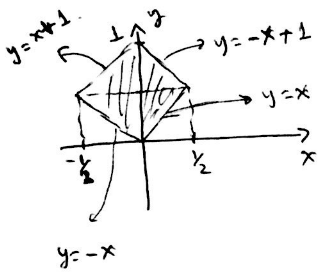
06) $\iint_B \sqrt[3]{y^2-x^2}\, dx\, dy$,

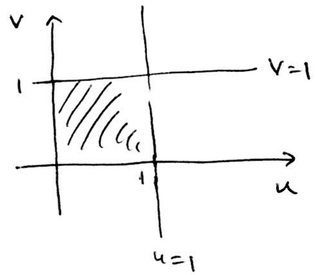
B é o paralelogramo de vértices $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, $(0,1)$, $(-\frac{1}{2},\frac{1}{2})$.

<u>Dem</u>: $y^2-x^2=(y-x)(y+x)$.

Mudança de variável:

$$\begin{cases} u = y-x \\ v = y+x \end{cases} \iff \begin{cases} x = \frac{v}{2}-\frac{u}{2} \\ y = \frac{v}{2}+\frac{u}{2} \end{cases}$$





$$-\ \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = -\frac{1}{2}.$$

$$\therefore\ dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \implies dx\,dy = \frac{1}{2}\, du\,dv.$$

Seja ψ a transformação dada por $\begin{cases} u = y-x \\ v = y+x. \end{cases}$,

$\psi(x,y)=(u,v)$

- ψ é a inversa da transformação $\varphi(u,v)=(x,y)$, def. por $\begin{cases} x = \frac{v}{2}-\frac{u}{2} \\ y = \frac{v}{2}+\frac{u}{2} \end{cases}$, observe que φ é de classe C^1.

– φ leva as retas $y=x$, $y=-x+1$, $y=x+1$ e $y=-x$, respectivamente, nas retas $u=0$, $v=1$, $u=1$ e $v=0$.

Segue:

$$\iint_B \sqrt[3]{y^2-x^2}\,dx\,dy = \int_0^1 \left[\int_0^1 \sqrt[3]{\left(\frac{v}{2}+\frac{u}{2}\right)^2 - \left(\frac{v}{2}-\frac{u}{2}\right)^2}\ \frac{1}{2}\,du \right] dv =$$

$$= \frac{1}{2} \int_0^1 \left[\int_0^1 \sqrt[3]{uv}\ du \right] dv = \cdots = \frac{9}{32}$$