P1 - FVV - Prof. Edson Iwaki - 12/03/2015

Nome:

1. Com relação a função $f : \mathbb{R}^2 \to \mathbb{R}$ definida por:

$$f(x,y) = \begin{cases} \frac{x^3 sen(x^2y)}{x^2+y^2} + 2x + 3y^2, & (x,y) \neq (0,0) \\ 0, & (x,y) \neq (0,0) \end{cases}$$

   (a) f é contínua em $(0,0)$?
   (b) f admite derivadas parciais em $(0,0)$? Em caso afirmativo, determine $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$.
   (c) As derivadas parciais $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas em $\mathbb{R}^2$?
   (d) f admite derivadas direcionais em $(0,0)$?
   (e) f é diferenciável em $(0,0)$?
   (f) f admite plano tangente em $(0,0)$? Determine-o.

2. Defina quando uma função $f : \mathbb{R}^3 \to \mathbb{R}$ é diferenciável no ponto (x_0, y_0, z_0).

3. Seja $F(x,y,z) = f(\frac{x}{y}, \frac{y}{z}, \frac{z}{x})$. Sabendo que a função $f : \mathbb{R}^3 \to \mathbb{R}$ é diferenciável, mostre que $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = 0, \forall (x,y,z) \neq (0,0,0)$.

Theorem 0.1 *(Regra da Cadeia) Sejam $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ funções. Seja $w = f(u,v,t)$, com $u = g(x,y,z), v = h(x,y,z), t = (x,y,z)$. Se f, g, h, t forem diferenciáveis, então:*

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x},$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y},$$

e

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z},$$

GABARITO P1.

Q1: Seja $f(x,y)$ definida por:

$$f(x,y) = \begin{cases} \dfrac{x^3 \operatorname{sen}(x^2 y)}{x^2+y^2} + 2x + 3y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) f é contínua em (0,0)?

SIM.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \overset{\to 0}{x} \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\text{Limitada}} \cdot \overset{\to 0}{\operatorname{sen}(x^2 y)} + \overset{\to 0}{2x} + \overset{\to 0}{3y^2} = 0 = f(0,0)$$

pq?

Como $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, f é contínua em (0,0).

b) f admite derivadas parciais em (0,0)?

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - \overset{=0}{f(0,0)}}{h} = \lim_{h\to 0} \frac{\frac{h^3 \operatorname{sen}(h^2 \cdot 0)}{h^2} + 2h}{h} = 2.$$

$$\therefore \boxed{\frac{\partial f}{\partial x}(0,0) = 2}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - \overset{=0}{f(0,0)}}{k} = \lim_{k\to 0} \frac{3k^2}{k} = 0$$

$$\therefore \boxed{\frac{\partial f}{\partial y}(0,0) = 0}$$

(c) As derivadas parciais $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas em $\mathbb{R}^2$?

$$\frac{\partial f}{\partial x} = \frac{\left(3x^2 \cdot sen(x^2y) + x^3 \cos(x^2y) \cdot 2xy\right)(x^2+y^2) - x^3 sen(x^2y) \cdot 2x}{(x^2+y^2)^2} + 2$$

$\forall (x,y) \neq (0,0)$

$$\frac{\partial f}{\partial y} = \frac{\left(x^3 \cos(x^2y) \cdot x^2\right)(x^2+y^2) - x^3 sen(x^2y) \cdot 2y}{(x^2+y^2)^2} + 6y$$

$\forall (x,y) \neq (0,0)$.

$\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas, $\forall (x,y) \neq (0,0)$,

pois são quocientes de funções polinomiais e portanto, são contínuas.

∴ Devemos verificar se $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas em $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{(x,y)\to(0,0)} \Bigg(3 \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\substack{\text{Limitada} \\ \text{pq?}}} sen(x^2y) + \overset{\to 0}{2x^2} \cdot \overbrace{\frac{x^2}{x^2+y^2}}^{\text{Limitada}} \cdot \overset{\to 0}{y} \cdot \overset{\to 1}{\cos(x^2y)} - \underbrace{\overbrace{\frac{2x^4}{(x^2+y^2)^2}}^{\text{Limitada}}}_{\text{pq?}} \overset{\to 0}{sen(x^2y)} + 2 \Bigg) = \underset{\substack{\uparrow \\ 1(b).}}{2} = \frac{\partial f}{\partial x}(0,0)$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y} = \lim_{(x,y)\to(0,0)} \overset{\to 0}{x^3} \cdot \underbrace{\overbrace{\frac{x^4}{(x^2+y^2)}}^{\text{Limitada}}}_{\text{pq}} \cdot \overset{\to 1}{\cos(x^2y)} - 2 \underbrace{\overbrace{\frac{x^3y}{(x^2+y^2)^2}}^{\text{Limitada}}}_{\text{pq?}} \overset{\to 0}{sen(x^2y)} = \underset{\substack{\| \, 1b \\ \frac{\partial f}{\partial y}(0,0)}}{0}$$

* $\forall (x,y) \neq (0,0)$

$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$

$|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$

$$\therefore \left| \frac{x^2 \; xy}{(x^2+y^2)^2} \right| \leq \left| \frac{x^2 \cdot \sqrt{x^2+y^2} \cdot \sqrt{x^2+y^2}}{(x^2+y^2)^2} \right| = \overbrace{\frac{x^2}{x^2+y^2}}^{\text{Limitada}} \leq 1.$$

$\therefore \frac{x^3y}{(x^2+y^2)^2}$ é limitada $\forall (x,y) \neq (0,0)$ *

1. $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas em $(0,0)$.

$\therefore$ $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas em $\mathbb{R}^2$.

(i.e. $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ são contínuas, $\forall (x,y) \in \mathbb{R}^2$.

01) d) f admite derivadas direcionais em $(0,0)$?

Sim, pois f é diferenciável em $(0,0)$ (por 1e)

e portanto, possui derivada direcional em $(0,0)$

na direção de qualquer vetor $\vec{u}$, unitário e
$\vec{u} = (h,k)$

e $\frac{\partial f}{\partial \vec{u}}(0,0) = \nabla f(0,0) \cdot \vec{u} = (2,0) \cdot (h,k) = \underline{2h}$.

01) e) f é diferenciável em $(0,0)$?

Sim, pois f possui derivadas parciais contínuas em $(0,0)$
por 1c.

<u>Solução 2</u>

$$\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - \overset{0}{f(0,0)} - 2h - 0k}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k)\to(0,0)} \frac{\frac{h^3 \operatorname{sen}(h^2k)}{h^2+k^2} + \cancel{2h} + 3k^2 - \cancel{2h}}{\sqrt{h^2+k^2}} =$$

$$= \lim_{(h,k)\to(0,0)} \underbrace{\frac{h}{\sqrt{h^2+k^2}}}_{\text{Limitada}} \cdot \underbrace{\frac{h^2}{h^2+k^2}}_{\text{limitada}} \cdot \underbrace{\operatorname{sen}(h^2k)}_{\to 0} + \underbrace{3k}_{\to 0} \cdot \underbrace{\frac{k}{\sqrt{h^2+k^2}}}_{\text{Limitada}} = 0$$

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$\therefore$ f é diferenciável em $(0,0)$, pois $\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\|(h,k)\|} = 0$,

onde $E(h,k) = f(0+h, 0+k) - f(0,0) - \frac{\partial f}{\partial x}(0,0)h - \frac{\partial f}{\partial y}(0,0)k$.

<u>Ver outra folha.</u>

$\forall (h,k) \neq (0,0)$

$|h| = \sqrt{h^2} \leq \sqrt{h^2+k^2} \Rightarrow \dfrac{|h|}{\sqrt{h^2+k^2}}$ é limitada.

$|k| = \sqrt{k^2} \leq \sqrt{h^2+k^2} \Rightarrow \dfrac{|k|}{\sqrt{h^2+k^2}}$ é limitada.

#

01) f. f admite plano tangente em $(0,0)$? Determine-o.

Sim. pois f é diferenciável em $(0,0)$.

Equação do plano tangente ao gráfico de f no ponto $(0,0,f(0,0))$

é dado por:

$$z - \underbrace{f(0,0)}_{=0} = \underbrace{\frac{\partial f}{\partial x}(0,0)}_{=2}(x-0) + \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0}(y-0)$$

$\therefore\ z = 2x.$

02) $f: \mathbb{R}^3 \to \mathbb{R}$ é diferenciável no ponto (x_0, y_0, z_0)

se existem $a, b, c \in \mathbb{R}$ tais que

$\lim\limits_{(h,k,l)\to(0,0,0)} \dfrac{E(h,k,l)}{\|(h,k,l)\|}$ existe e $\lim\limits_{(h,k,l)\to(0,0,0)} \dfrac{E(h,k,l)}{\|(h,k,l)\|} = 0$,

onde $E(h,k,l) = f(x_0+h, y_0+k, z_0+l) - f(x_0,y_0,z_0) - ah - bk - cl$

03) Seja $F(x,y,z) = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$. Sabendo que a função $f: \mathbb{R}^3 \to \mathbb{R}$ é diferenciável. Mostre que

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = 0, \quad \forall (x,y,z) \neq (0,0,0).$$

Dem: $u = \frac{x}{y}$, $v = \frac{y}{z}$, $w = \frac{z}{x}$

$$\underbrace{(x,y,z) \longmapsto (u,v,w) \longmapsto f(u,v,w)}_{F}$$

Regra da Cadeia:

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{1}{y} \frac{\partial f}{\partial u} - \frac{z}{x^2} \frac{\partial f}{\partial w} \quad ①$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y} = -\frac{x}{y^2} \frac{\partial f}{\partial u} + \frac{1}{z} \frac{\partial f}{\partial v} \quad ②$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z} = -\frac{y}{z^2} \frac{\partial f}{\partial v} + \frac{1}{x} \frac{\partial f}{\partial w} \quad ③$$

De ①, ② e ③:

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} =$$

$$= \frac{x}{y} \frac{\partial f}{\partial u} - \frac{z}{x} \frac{\partial f}{\partial u} - \frac{x}{y} \frac{\partial f}{\partial u}$$

$$+ \frac{y}{z} \frac{\partial f}{\partial v} - \frac{y}{z} \frac{\partial f}{\partial v} + \frac{z}{x} \frac{\partial f}{\partial w} = 0$$

$\square$

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