1st Exam - Analytic Geometry A1 - Professor: Juliana Pimentel - 20.10.2017

NAME: **RA:** **GRADE:**

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1. (2.5 points) Prove that the diagonals of a parallelogram have the same midpoint.

2. (2.5 points) Consider the vectors $\vec{u} = (2, -1, 2)$ and $\vec{v} = (1, 2, -2)$. Find scalars a, b such that $\vec{w} = a\vec{u} + b\vec{v}$ and $\vec{w} \cdot \vec{v} = 0$.

3. (2.5 points) Let $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be a basis. Consider the vectors $\vec{f}_1 = \vec{e}_1 - \vec{e}_2 - \vec{e}_3$, $\vec{f}_2 = \vec{e}_1 + 2\vec{e}_2 + \vec{e}_3$ and $\vec{f}_3 = 2\vec{e}_1 + \vec{e}_2 + 4\vec{e}_3$.

$\vec{f}_1$ $\vec{f}_2$ $\vec{f}_3$

(a) Determine whether or not $B' = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a new basis.

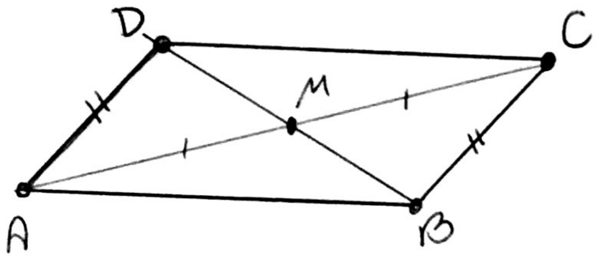
(b) If yes, find the coordinates of $\vec{v} = 3\vec{e}_1 - 5\vec{e}_2 + 4\vec{e}_3$ with respect to B'.

4. (2.5 points) Let O, A, B, C be vertices of a regular tetrahedron with edge length 1. Find the angle between the vectors $\vec{OP} = \vec{OA} + 2\vec{OB} - \vec{OC}$ e $\vec{OQ} = \vec{OA} + \vec{OB} + 2\vec{OC}$.

GOOD LUCK!

1st Exam - Analytic Geometry - A1

1st Question



Suppose that M is the midpoint of $\vec{AC}$. We want to prove that M is also the midpoint of $\vec{DB}$. More precisely, we will prove that

$$\vec{BM} = \vec{MD}.$$

Indeed we have

$$\vec{BM} = \vec{BC} + \vec{CM} = \vec{AD} + \vec{MA} = \vec{MA} + \vec{AD} = \vec{MD}.$$

2nd Question

$$\vec{\omega} = a\vec{u} + b\vec{v} = a(2,-1,2) + b(1,2,-2) = (2a+b, -a+2b, 2a-2b)$$

$$\vec{\omega} \cdot \vec{v} = 0 \Rightarrow (2a+b, -a+2b, 2a-2b) \cdot (1, 2, -2) = 0$$

$$\Rightarrow 1.(2a+b) + 2(-a+2b) - 2(2a-2b) = 0$$

$$\Rightarrow \cancel{2a} + b - \cancel{2a} + 4b - 4a + 4b = 0$$

$$\Rightarrow 9b - 4a = 0$$

$$\Rightarrow b = \frac{4a}{9}$$

We may choose any scalars a and b satisfying

$$\boxed{b = \frac{4a}{9}}$$

3rd Question

(a) B' is indeed a new basis since we have

$$f_1 = (1, -1, -1)_B$$
$$f_2 = (1, 2, 1)_B$$
$$f_3 = (2, 1, 4)_B$$

and, therefore,

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{vmatrix} = -1 \neq 0.$$

(b) We need to find scalars a, b, c such that

$$\vec{v} = a\vec{f_1} + b\vec{f_2} + c\vec{f_3}.$$

In B coordinates we have

$$\vec{v} = (3, -5, 4)_B = a(1,-1,-1)_B + b(1,2,1)_B + c(2,1,4)_B.$$

Then

$$\begin{cases} 3 = a + b + 2c & \text{II} \\ -5 = -a + 2b + c & \text{I} \\ 4 = -a + b + 4c \Rightarrow a = b + 4c - 4 \overset{\text{I}}{\Rightarrow} -5 = -(b+4c-4) + 2b + c \end{cases}$$

$$\Rightarrow -5 = b - 3c + 4 \Rightarrow b = -5 + 3c - 4$$

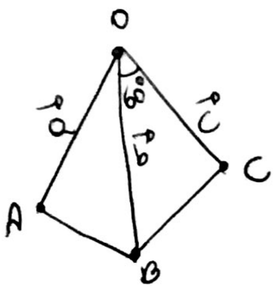
$$\overset{\text{II}}{\Rightarrow} 3 = (b + 4c - 4) + b + 2c = 2b + 6c - 4$$

$$\Rightarrow 3 = 2(-5 + 3c - 4) + 6c - 4 = -22 + 12c \Rightarrow \boxed{c = \frac{25}{12}}$$

$$\Rightarrow \boxed{b = -\frac{11}{4}, \; a = +\frac{19}{12}}$$

4th Question

We write $\vec{OP}=\vec{p}$, $\vec{OQ}=\vec{q}$, $\vec{OA}=\vec{a}$, $\vec{OB}=\vec{b}$ and $\vec{OC}=\vec{c}$.



Then, the angle between $\vec{p}$ and $\vec{q}$ satisfies $\cos\theta = \dfrac{\vec{p}\cdot\vec{q}}{|\vec{p}|\,|\vec{q}|}$. First notice that

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos 60° = 1\cdot 1\cdot \frac{1}{2} = \frac{1}{2} = \vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}.$$

It follows that

$*$ $$\begin{aligned}\vec{p}\cdot\vec{q} &= (\vec{a}+2\vec{b}-\vec{c})\cdot(\vec{a}+\vec{b}+2\vec{c})\\ &= |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot 2\vec{c}+2\vec{b}\cdot\vec{a}+2|\vec{b}|^2+4\,\vec{b}\cdot\vec{c}\\ &\quad -\vec{c}\cdot\vec{a}-\vec{c}\cdot\vec{b}-2|\vec{c}|^2\\ &= 1+\frac{1}{2}+2\cdot\frac{1}{2}+2\cdot\frac{1}{2}+2+4\cdot\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-2=\frac{9}{2}\end{aligned}$$

$*$ $$\begin{aligned}|\vec{p}|^2 &= (\vec{a}+2\vec{b}-\vec{c})\cdot(\vec{a}+2\vec{b}-\vec{c})\\ &= |\vec{a}|^2+2\,\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}+2\vec{b}\cdot\vec{a}+4|\vec{b}|^2-2\vec{b}\cdot\vec{c}\\ &\quad -\vec{c}\cdot\vec{a}-2\,\vec{c}\cdot\vec{b}+|\vec{c}|^2\\ &= 1+2\cdot\frac{1}{2}-\frac{1}{2}+2\cdot\frac{1}{2}+4\cdot 1-2\cdot\frac{1}{2}-\frac{1}{2}-2\cdot\frac{1}{2}+1=5\end{aligned}$$

$*$ $$\begin{aligned}|\vec{q}|^2 &= (\vec{a}+\vec{b}+2\vec{c})\cdot(\vec{a}+\vec{b}+2\vec{c})\\ &= |\vec{a}|^2+\vec{a}\cdot\vec{b}+2\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+|\vec{b}|^2+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}\\ &\quad +2\vec{c}\cdot\vec{b}+4|\vec{c}|^2\\ &= 1+\frac{1}{2}+2\cdot\frac{1}{2}+\frac{1}{2}+1+2\cdot\frac{1}{2}+2\cdot\frac{1}{2}+2\cdot\frac{1}{2}+4\cdot 1=11\end{aligned}$$

Therefore,

$$\cos\theta = \frac{9/2}{\sqrt{5}\,\sqrt{11}}, \quad \boxed{\theta = \arccos\left(\frac{9}{2\sqrt{5}\sqrt{11}}\right)}.$$