UNIVERSIDADE DE LISBOA

INSTITUTO SUPERIOR TÉCNICO

Asymptotic behavior of slowly non-dissipative systems

Juliana Fernandes da Silva Pimentel

Supervisor: Doctor Carlos Alberto Varelas da Rocha

Thesis approved in public session to obtain the PhD degree in Mathematics

Jury final classification: Pass with Merit

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To Edgard

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Resumo

Consideramos sistemas dinâmicos gerados por equações escalares de reação-difusão. Em geral, esperamos que estes sejam dissipativos, rapidamente não-dissipativos ou lentamente não-dissipativos. Isto significa, respetivamente, que as soluções são todas eventualmente limitadas, os sistemas exibem *blow-up* em tempo finito ou existem soluções cujas normas crescem ao infinito com o tempo. Apesar de haver substancial quantidade de informações acerca das duas primeiras classes de sistemas, apenas recentemente a última classe tem sido considerada. Nesta tese estudamos sistemas lentamente não-dissipativos gerados por equações escalares de reação-difusão com não-linearidade dependendo da variável espacial e a depender de um termo de advecção.

Ao se considerarem sistemas lentamente não-dissipativos, a existência de soluções não limitadas - às quais nos referimos como soluções do tipo *grow-up* - é necessária a introdução de certos objectos no infinito, interpretados como equilíbrios no infinito. Mais ainda, a existência destas soluções introduz uma estrutura de órbitas mais complexa no atrator do que aquela gerada por sistema dissipativos. As soluções do tipo *grow-up* nas variedades instáveis dos equilíbrios limitados são definidas como conexões heteroclínicas com os equilíbrios no infinito e, a partir da teoria de variedades inerciais, determinamos os limites exatos das soluções do tipo *grow-up*. Então, estendendo resultados anteriores, obtemos a existência de um atrator global não compacto, composto pela reunião do conjunto dos equilíbrios limitados, o conjunto dos equilíbrios no infinito e as conexões heteroclínicas entre eles.

É sabido que existe uma permutação associada a sistemas dissipativos que determina grande parte da geometria do atrator global. Para sistemas não-dissipativos, a existência de equilíbrios no infinito engendra dificuldades substanciais na obtenção de uma permutação semelhante determinando as conexões heteroclínicas no atrator não compacto. Neste contexto, ainda conseguimos determinar as conexões heteroclínicas com base no método da permutação de Sturm. Esta técnica oferece um critério mais simples para descrever o atrator não compacto e generaliza resultados anteriormente obtidos para equações dissipativas.

Palavras-chave: sistemas lentamente não dissipativos, soluções do tipo *grow-up*, número de zeros, propriedades nodais, y-*map*, atrator global, conexões heteroclínicas, variedade inercial, suspensão, permutação Sturm.

Abstract

We consider dynamical systems generated by scalar reaction-diffusion equations. Generally speaking we expect it to be dissipative, fast non-dissipative or slowly non-dissipative. This means, respectively, that the solutions are all ultimately bounded, the system exhibits finite-time blow-up or there exists solutions whose norms grow-up to infinity with time. Although there is a great deal of information regarding the first two classes of dynamical systems, only recently the latter class has been approached. In this thesis we address slowly non-dissipative systems generated by scalar reaction-diffusion equations with the nonlinearity depending on the space variable and possessing an advection term.

When dealing with slowly non-dissipative systems, the existence of unbounded solutions, which are referred to as grow-up solutions, requires the introduction of some objects at infinity interpreted as equilibria at infinity. Moreover, the existence of these solutions yields a more complex orbit structure on the attractor than that appearing on dissipative systems. The grow-up solutions in the unstable manifolds of bounded equilibria are defined to be heteroclinic connections with the equilibria at infinity and, by recurring to the theory of inertial manifolds, we are able to determine the exact limits of the grow-up solutions. Then, by extending known results, we obtain the existence of a non-compact global attractor which is composed of the set of bounded equilibria, the set of equilibria at infinity and the heteroclinic connections between them.

It is well known that there exists a permutation associated with dissipative systems that determines many of the main geometric features of the global attractor. For non-dissipative systems, the existence of equilibria at infinity adds some significant challenges to obtain a similar permutation determining the heteroclinic connections on the non-compact global attractor. Under this setting, we still manage to determine the heteroclinic connections based on the Sturm permutation method. This provides a simple criterion for describing the non-compact global attractor and generalize the results obtained for dissipative equation.

Keywords: slowly non-dissipative systems, grow-up solutions, zero number, nodal properties, y-map, global attractor, heteroclinic connections, inertial manifold, suspension, Sturm permutation.

Contents

Chapter 1

Introduction

A better understanding of longtime dynamical properties of infinite dynamical systems generated by scalar parabolic PDE's is of paramount relevance. A primary case approached in the last decades involves a crucial assumption of existence of a bounded set in the state space into which every solution eventually enters and remains. Such a set is known as the global attractor. Dynamical systems possessing this property are known as dissipative and a well established theory concerning their asymptotic behavior is composed of a large number of results derived, for instance, in [Hal88], [BV92] and [Lad91]. Meanwhile, motivated by numerous applications, a different class of equations has been approached. It comprises equations which generate a dynamical system not containing an uniformly bounded attracting set, see [Sch07, BG11b, BG11c]. Particular attention has been paid to systems with a subset of solutions blowing-up at infinite time, [BG10]. The dissipativity property of those systems is obviously no longer verified.

Generally speaking, we expect the solutions of a scalar reaction-diffusion equation to present one of the following possible behaviors: the induced dynamical system will be ultimately bounded, for at least one initial condition the solution will experience a finite-time blow-up or finite-time blow-up does not take place but a subset of solutions present infinite-time blow-up. The usual denomination referring to any of these types of equations is dissipative, fast non-dissipative and slowly non-dissipative, respectively. Despite the fact that a devoted attention has already been paid to the first two categories, just recently the latter case has been approached, [BG10], and it is precisely our object of study.

Consider a general scalar parabolic partial differential equation of the form

$$\begin{cases} u_t = u_{xx} + f(x,u,u_x) \\ u(0,.) = u_0, \end{cases} \tag{1.1}$$

with x defined on a bounded interval and satisfying separated boundary conditions. An example of an explicit condition to impose on f for the equation to generate a dissipative system is

$$f(x,u,0).u < 0, \tag{1.2}$$

for $|u|$ large enough, and moreover requiring

$$|f(x,u,p)| \leq c(1+|p|^{\gamma})$$

with $c > 0$ and $0 \leq \gamma < 2$, uniformly for x and u in compact sets, [Ama85]. These conditions motivate us to consider a specific nonlinearity in order to obtain systems which are not dissipative. We can take for instance a function of the form

$$f(x,u,u_x) = bu + g(x,u,u_x), \tag{1.3}$$

with $b > 0$, to jeopardize the dissipativity of equation (1). Furthermore, for g bounded we can guarantee that the solutions of equation (1) exist for all forward time and the equation that we are considering will, in fact, produce the behavior arising in slowly non-dissipative equations. But notice that if g is bounded and $b < 0$ we are still provided with a dissipative system.

In the study of dynamical systems generated by equation (1), for a class of nonlinearities f, the global attractors have been proven to be of paramount relevance. For dissipative equations such attractors are non-empty, compact, invariant attracting sets in some appropriate underlying space X. Their characterization turned out to be a crucial tool for describing the asymptotic behavior of such systems, as we can see in [BF88, BF89, FR96, Bru90, Roc91]. However, when dealing with slowly non-dissipative systems, we are no longer provided with the existence of a bounded set in X attracting each bounded subset of X. In this way, for the case where all solutions are guaranteed to exist for all forward time, although we still have a minimal invariant set attracting all solutions, we cannot ensure compactness. Such an invariant set still provides us with relevant information concerning the asymptotic behavior of solutions for equation (1). This object will be the center of our investigation and we shall designate it as non-compact global attractor.

In this thesis we aim to address the characterization of the non-compact global attractor for the equation (1) in the interval $[0,\pi]$ under Neumann boundary conditions, that is,

$$\begin{cases} u_t = u_{xx} + f(x,u,u_x), \ x \in (0,\pi) \\ u_x(t,0) = u_x(t,\pi) = 0, \end{cases} \tag{1.4}$$

with $f(x,u,u_x) = bu + g(x,u,u_x)$. The objective is the extension of certain results that hold when the nonlinearity is dissipative to the slowly non-dissipative case. In addition, we consider a more general nonlinearity than that addressed in [BG11b, BG11c], in the sense that we allow it to exhibit dependence on the space variable and to possess an advection term.

Despite the non-dissipativity of equation (1.4) we obtain a Lyapunov functional for the generated dynamical system and we obtain as a consequence that the solutions that do not converge to any bounded equilibria can not remain bounded. Moreover, we refer to these unbounded solutions as grow-up solutions. From the techniques obtained in [Hel11], related to a Poincaré projection, one derives the existence of some objects at infinity interpreted as equilibria at infinity. In order to determine the exact limit of the grow-up solutions, we recur to the theory of inertial manifold since the L^2-norm alone does not prevent the zero number to decay at $t = \infty$. We are then able to obtain the transfinite heteroclinics by Infinite Blocking and Infinite Liberalism notion as in the dissipative realm. More precisely, given a bounded equilibrium and an equilibrium at infinity, the Infinite Blocking Lemma provides sufficient conditions for a connection between the given equilibria to be blocked and the Infinite Liberalism Lemma states that connections between the equilibria exist whenever they are not blocked. The intra-infinite heteroclinics are derived as in [Hel11].

When dealing with dissipative equations, we have an associated permutation that determines many of the main geometric features of the global attractor, as we can see in [FR91, FR96, Wol02]. One of our main goals is to obtain a permutation related to the slowly non-dissipative equation (1.4) that determines the heteroclinic connections on the non-compact global attractor. In this way we obtain a simple criterion for describing the non-compact global attractor and generalize the results obtained for dissipative equation.

To obtain such a permutation we proceed as follows. We obtain that the non-compact global attractor $\mathcal{A}_f$ is composed of a bounded subset $\mathcal{A}_f^c \subset B$, for some large ball $B \subset X^\alpha$, and an unbounded subset $\mathcal{A}_f^\infty$. We define the permutation σ_f associated to the slowly non-dissipative equation (1.4) as in the dissipative case, by labeling the equilibria ordered firstly by their values at $x = 0$ and then at $x = \pi$. We then consider the k-th suspension of σ_f, where $k = [\sqrt{b}] + 1$. The obtained permutation $\hat{\sigma}_f^1$ is proved to be Sturm, which implies that it is realizable. It is then obtained a dissipative problem of the form

$$\begin{cases} u_t = u_{xx} + h(x, u, u_x), \ x \in (0, \pi) \\ u_x(t, 0) = u_x(t, \pi) = 0, \end{cases} \tag{1.5}$$

with the associated permutation σ_h defined in the usual way for dissipative equations coinciding with $\hat{\sigma}_f^1$. Since equation (1.5) is dissipative, we have a decomposition of the global attractor in terms of the adjacency notion, as obtained in [Wol02]. Moreover, we construct the function h in such a way that it coincides with f in the large ball $B \subset X^\alpha$ and is dissipative, that is, $h(x, u, u_x) = cu$ for some $c < 0$, outside a larger ball $\hat{B}$. Given that, the equilibria of equation (1.5) that are contained in B coincide with the bounded equilibria of the non-dissipative equation (1.4). We then make a correspondence between the remaining equilibria of equation (1.5) with the equilibria at infinity. The correspondence preserves the heteroclinic connections between the equilibria and, as a result, we obtain a simple criterion to determine the connecting orbit structure on the non-compact global attractor using the idea of adjacency.

This thesis is organized as follows. In Chapter 2 we focus on the asymptotic behavior of the grow-up solutions. We recall a crucial tool developed in [Hel11] to get a better understanding of the behavior of the grow-up solutions at infinity. Then the concept of equilibria at infinity is introduced.

In Chapter 3 we discuss some standard tools which are frequently used in the study of the connecting orbit structure of attractors. We recall the definition and some crucial properties of the zero number. We also recall the definition of the functional known as y-map which provides information on the zero number of solutions converging in backwards to an equilibrium. We also reproduce the discussion in [BG11c], where an extended form of the y-map is obtained to include applications to non-dissipative equations and it is proved that the extended y-map is surjective.

In Chapter 4 we draw our attention to the decomposition of the non-compact global attractor. We first obtain the existence of an unbounded inertial manifold containing the non-compact global attractor, from the results obtained in [Mik91]. We then present a discussion on the grow-up solutions in the unstable manifold of the equilibria, since these are interpreted as a connection from a bounded equilibrium to an equilibrium at infinity. We obtain the Infinite Blocking Liberalism Lemma and the Infinite Liberalism Lemma, and therefore, the transfinite heteroclinics. A description of the heteroclinics within infinity is also presented. We then derive some results for the bounded equilibria in order to obtain a description of the bounded heteroclinics in the next chapter. Let $E_f^c = \{v_1, ..., v_n\}$ be the set of bounded equilibria ordered by their values at $x = 0$. We obtain the non-emptiness of the set E_f^c and calculate the Morse indices of the equilibria v_1 and v_n.

In Chapter 5 we describe the bounded heteroclinics and obtain a simple criterion to describe the heteroclinic connections on the non-compact global attractor in terms of the adjacency notion. We define a suspension of any given meander permutation and by considering suspensions of the permutation σ_f related to equation (1.4), we obtain a dissipative system that is associated to the original non-dissipative equation (1.4). We then present our main result, Theorem 5.4.1, describing the heteroclinic connections in terms of the adjacency notion. The obtained Theorem generalizes the main result in [BG10] and also provides a simple criterion for describing the connections.

Chapter 2

Non-dissipativity

2.1 Reaction-diffusion equation

We consider the following scalar reaction-diffusion equation

$$\begin{cases} u_t = u_{xx} + f(x, u, u_x), \quad x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0. \end{cases} \tag{2.1}$$

where $f(x, u, u_x) = bu + g(x, u, u_x)$, $b > 0$ and $g : [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}$ is a bounded C^2 function. We also assume that $g(x, u, p)$ is globally Lipschitz in (u, p). This implies, from standard theory (see, for instance, [Ama85, Hal88, Hen81]) that we are provided with a local solution semigroup defined by $u(t, .) = S(t)u_0$ for $t \geq 0$ and the initial condition u_0 in the underlying space. We take, in the case of equation (2.1), the Hilbert space

$$X = L^2([0, \pi]).$$

We denote the norm in X by $\| \cdot \|$. Moreover, we denote by A the operator $-\partial_{xx} - bI$. We know that A is a sectorial operator in X and $-A$ generates the analytic semigroup e^{-tA}, [Hen81, Paz83]. In fact, for $b < 0$, A is a positive definite selfadjoint operator. In any case, for

$$A_1 = A + (b + 1)I,$$

the fractional power spaces

$$X^\alpha := D((A_1)^\alpha),$$

for each $\alpha \geq 0$, are well defined with the graph norm $\|x\|_\alpha := \|(A_1)^\alpha x\|$, $x \in X^\alpha$. It is worth notice that, if $b < 0$ then we can choose $A_1 := A$, as the eigenvalues of A with Neumann boundary conditions in this case are all positive. On X^α, with $\alpha > \frac{3}{4}$, the Nemitskii operator of g defined by

$$G(u)(x) = g(x, u, u_x),$$

takes values in X and is globally Lipschitz in u. We chose $\alpha > \frac{3}{4}$ due to the fact that, in this case, $X^\alpha \subset C^1$. The solution semigroup $S(t)$ is defined in the underlying space X^α for $t \geq 0$,

$$S(t) : X^\alpha \longrightarrow X^\alpha.$$

Before we proceed it is important to make precise the definitions of dissipative, fast non-dissipative and slowly non-dissipative dynamical systems.

The equation (2.1) generates a dissipative dynamical system if there exists a fixed large ball B in the underlying space X^α such that any solution

$$u(t,.) = S(t)u_0,$$

with initial condition u_0, will enter and remain within B for all time $t \geq T(u_0)$. A dynamical system is referred to as fast non-dissipative if there exists an initial condition u_0 such that the maximal time $t(u_0)$ of existence and uniqueness of the corresponding solution is finite, that is, $t(u_0) < \infty$. Closing the gap between the former classes of systems are those which were recently introduced and are called slowly non-dissipative systems. It comprises all of those such that global existence and uniqueness are guaranteed for every solution, that is, for all initial condition $u_0 \in X^\alpha$ the maximal time $t(u_0)$ such that $S(t)u_0$ exists is $t(u_0) = \infty$, but for at least one initial condition u_0 there does not exist any time $T(u_0)$ such that the solution can be bounded for all $t \geq T(u_0)$, in X^α.

We now introduce some usual terminology which will be useful for a better understanding of the contrast between the dynamical systems just defined. We say that a fast non-dissipative system presents finite-time blow-up, when we want to refer to the occurrence of a solution whose maximal time of existence and uniqueness is finite. Similarly, one uses the term infinite-time blow-up for the behavior arising in a slowly non-dissipative system when a solution is defined for all forward time but whose norm is unbounded. Such an unbounded solution will be referred to as a grow-up solution.

Under the above terminology, one should notice for instance that the existence of one solution experiencing finite-time blow-up is enough to make a dynamical system fast non-dissipative. Moreover, although slowly non-dissipative systems must exhibit blow-up, finite time blow-up cannot occur in such systems. Obviously, a dynamical system with the dissipativity property is not allowed to carry either a finite-time or an infinite-time blow-up.

Once we assume boundedness of g, the dynamical system generated by (2.1) is either dissipative or slowly non-dissipative. Indeed, for each $u_0 \in X^\alpha$ there is a unique solution defined on some maximal interval $0 \leq t < t(u_0)$, and we further know from [Hen81] that either $t(u_0) = +\infty$ or

$$\lim_{t \to t(u_0)} \|S(t)u_0\|_\alpha = \infty.$$

But, if $\|G(u)\| \leq \Gamma$ for all $u \in X^\alpha$, then

$$\begin{aligned}\|S(t)u_0\|_\alpha &= \|e^{-At}u_0 + \int_0^t e^{-A(t-s)}G(u)ds\|_\alpha \\ &\leq \|e^{-At}u_0\|_\alpha + \Gamma \int_0^t \|(A_1)^\alpha e^{-A(t-s)}\|ds \\ &\leq Ce^{bt}\|u_0\|_\alpha + \Gamma \int_0^t (t-s)^{-\alpha}e^{b(t-s)}ds,\end{aligned}$$

which is bounded for each $0 < t < \infty$, for $\alpha \in (3/4, 1)$. One thus concludes that $t(u_0) = +\infty$ for all $u_0 \in X^\alpha$. Then, since all solutions of (2.1) exist for all forward time, a finite-time blow-up does not take place and the dynamical system obtained is either dissipative or slowly non-dissipative.

The following lemma derived in [BG11b] ensures that a sufficient condition on equation (2.1) so that it generates a slowly non-dissipative system is that $b > 0$. We consider a basis $\{\varphi_j(x)\}_{j\in\mathbb{N}_0}$ that is orthonormal in $L^2([0,\pi])$, comprised of the eigenfunctions of the operator A with Neumann boundary conditions, i.e., $\varphi_j(x) = \sqrt{\frac{2}{\pi}}\cos jx$ for $j = 1, 2, ...$ and $\varphi_0(x) = \sqrt{\frac{1}{\pi}}$. We further denote by λ_j the corresponding eigenvalues and we observe that they are given by $\lambda_j = j^2 - b$, for each $j \in \mathbb{N}_0$.

Lemma 2.1.1. *If $b > 0$ then the dynamical system generated by equation (2.1) is non-dissipative.*

Proof. The solutions of (2.1) are defined for all $t \geq 0$, which implies that a finite-time blow-up can not occur. It remains then to prove the existence of at least one solution with infinite-time blow-up. For that we consider the eigenspaces E_j of A associated with each eigenvalue λ_j. We notice that any solution $u(t,x)$ of (2.1) can written as

$$u(t,x) = \sum_{j=0}^{\infty} \hat{u}_j(t)\varphi_j(x),$$

where $\hat{u}_j(t) = \langle u(t,.), \varphi_j(.)\rangle_{L^2}$ and $\langle .,.\rangle_{L^2}$ denotes the inner product in $L^2([0,\pi])$. We then project equation (2.1) onto E_j and notice that if $b > 0$ then

$$\lambda_j = j^2 - b < 0$$

at least for $j = 0$:

$$\begin{aligned}\frac{d}{dt}\hat{u}_j(t) &= \frac{d}{dt}\langle u(t,.), \varphi_j(.)\rangle_{L^2} = \langle u_t(t,.), \varphi_j(.)\rangle_{L^2} \\ &= \langle u_{xx} + bu, \varphi_j(.)\rangle_{L^2} + \langle g(x,u,u_x), \varphi_j(.)\rangle_{L^2},\end{aligned}$$

which, by selfadjointness of A, implies

$$\frac{d}{dt}\hat{u}_j(t) = -\lambda_j\hat{u}_j(t) + \langle g(x,u,u_x), \varphi_j(x)\rangle_{L^2}. \tag{2.2}$$

If we write

$$\langle g(x,u,u_x), \varphi_j(x)\rangle_{L^2} = \langle G(u)(x), \varphi_j(x)\rangle_{L^2},$$

where $G(u)$ is the Nemitskii operator for the function $g(u)$, and denote

$$\hat{g}_j(t) := \langle G(u(t,x))(x), \varphi_j(x) \rangle_{L^2},$$

we finally get the following equation by rewriting (2.2)

$$\frac{d}{dt}\hat{u}_j(t) = -\lambda_j \hat{u}_j(t) + \hat{g}_j(t). \tag{2.3}$$

We then have a linear non-homogeneous first order ODE and from standard theory we know that a general solution of (2.3) has the form

$$\hat{u}_j(t) = \hat{u}_j^p(t) + \hat{u}_j^h(t),$$

where $\hat{u}_j^p(t)$ is a particular solution for (2.3) and $\hat{u}_j^h(t)$ is the solution of the corresponding homogeneous equation. As a particular solution we take

$$\hat{u}_j^p(t) = \int_{\infty}^{t} e^{-\lambda_j(t-s)} \hat{g}_j(s) ds$$

and we obtain the solution of (2.3) with the condition

$$\hat{u}_j(0) = \int_{\infty}^{0} e^{\lambda_j s} \hat{g}_j(s) ds + \hat{u}_j^h(0),$$

which is given by

$$\hat{u}_j(t) = \hat{u}_j^h(0) e^{-\lambda_j t} + \int_{\infty}^{t} e^{-\lambda_j(t-s)} \hat{g}_j(s) ds.$$

At this point one should notice that $G(u)$ being bounded in L^2 implies that $\hat{g}_j(t)$ is uniformly bounded for all $j \in \mathbb{N}_0$. Consequently, the particular solution $\hat{u}_j^p(t)$ is uniformly bounded for all $j \in \mathbb{N}_0$ such that $\lambda_j < 0$. On the other hand, whenever λ_j is strictly negative, if $\hat{u}_j^h(0) \neq 0$, then

$$\hat{u}_j^h(t) = \hat{u}_j^h(0) e^{-\lambda_j t}$$

grows exponentially fast as t goes to infinity.

From the Fourier decomposition obtained from the projections onto the eigenspaces E_j, we conclude that the L^2-norm of a solution $u(t,x)$ for (2.3) is $\|u(t,x)\| = \sqrt{\sum\limits_{i=0}^{\infty} (\hat{u}_i(t))^2}$. Thus, since we have $\lambda_j < 0$ at least for $j = 0$, if we take an initial condition u_0 such that $\hat{u}_0^h(0) \neq 0$, then

$$\hat{u}_0(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty.$$

The corresponding solution $u(t,x)$ for such an initial condition exhibits infinite-time blow-up since its

L^2-norm grows to infinite with time. This concludes the proof. $\square$

Motivated by the proof of the above lemma, or specifically by the existence of at least one solution for equation (2.1) whose appropriate norm grows to infinity with time, we aim to investigate such a behavior. Recall that an unbounded solution in this way is a grow-up solution, and we shall henceforward use this term. We are concerned, in particular, with the initial conditions leading to a grow-up solution, as well as with the limit of those solutions. We shall start considering these matters in the next subsection.

2.2 Non-compact global attractor

When dealing with dynamical systems which are provided with the existence of a Lyapunov functional, one possesses a huge advantage in terms of describing the generated flow in the global attractor. Fortunately, despite the fact of equation (2.1) being slowly non-dissipative, we can still guarantee the existence of a Lyapunov functional and it is given in the form

$$V(u) = \int_0^\pi H(x, u, u_x) dx$$

where $H(x, u, p)$ is a smooth function on $[0, \pi] \times \mathbb{R} \times \mathbb{R}$ with $H_{pp} > 0$ (see [Zel68, Mat88]). The functional V satisfies

$$\frac{d}{dt} V(u) = -\int_0^\pi H_{pp}(x, u, u_x) u_t^2 dx \leq 0.$$

In order to begin the discussion regarding the existence of unbounded solutions for equation (2.1), we present the next lemma derived in [BG11b].

Lemma 2.2.1. *Consider the equation (2.1) with $b > 0$ and an initial condition $u_0 \in X^\alpha$. Then the corresponding solution $u(t, .)$ either converges to some bounded equilibrium as t goes to infinity or $u(t, .)$ cannot stay bounded in any subset of X^α for all time t, that is, given any ball B in X^α and $T > 0$ the solution will leave B at some time $t^* > T$.*

Proof. Suppose neither of the above settings is verified, which is equivalent to say that the solution does not converge to any bounded equilibrium and that it does exist a ball B and some time $T > 0$ such that the solution enters at $t = T$ and remains within B whenever $t > T$. If this is the case, there exists a sufficiently large ball B^* in X^α, depending on B and T, such that

$$u(t, .) \subset B^*, \text{ for all } t \geq 0.$$

Since the semigroup $S(t)$ obtained from equation (2.1) is compact and it is guaranteed the existence of a Lyapunov functional with the above mentioned properties, the LaSalle invariance principle, [Hal69], can be applied. From that we conclude that the orbit $u(t, .)$ through u_0 must converge to some equilibrium of (2.1) contained in B^*, which contradicts our assumption. $\square$

We must recall that a global attractor for equation (1.1) is a nonempty maximal compact invariant set attracting each bounded set in the appropriate state space. We have then noticed that the system induced by (2.1) possesses solutions which are not allowed to remain bounded for all time. One of the immediate consequences is that we are no longer provided with a maximal compact invariant set, and therefore it does not exist a global attractor for the generated semigroup $S(t)$.

In the meantime, we know that the existence of a global attractor plays a crucial role regarding a better understanding of the long-time dynamics of equation (1.1). Taking this into account, it makes sense to overcome the non-boundedness induced by (2.1), and define a global attractor which is not compact. The following definition was introduced in [BG11b].

Definition 2.2.1. *A non-compact global attractor for the equation (2.1) is a non-empty minimal set in X^α attracting all bounded sets of X^α.*

When considering the set of all solutions with initial condition u_0 that remain bounded for all time t, then we are guaranteed the existence of a large ball B in X^α containing all of such solutions. In this way, as we have observed before, we are allowed to apply the theory used for the dissipative case, in particular taking advantage of the existence of a Lyapunov functional, and consequently distinguishing between the bounded subset of the non-compact global attractor and non-bounded one. We shall later confirm that this is in fact the case, since we do not have all the required machinery yet. It remains then to characterize the unbounded subset of the non-compact global attractor. In order to do that we shall consider the grow-up solutions and their respective limits. It should be stressed at this point that the reason why we call the equilibria of (2.1) bounded equilibria is that we are anticipating the existence of objects at infinity referred to as limits of the grow-up solutions. Such objects will be referred to as unbounded equilibria. It is convenient to pass now to an investigation of these objects.

2.3 Unbounded equilibria

We proceed as in [BG11b] and consider the basis $\{\varphi_j\}_{j\in\mathbb{N}_0}$ which is orthonormal in $L^2([0,\pi])$. When we assume boundedness of g by some scalar Γ, i.e.,

$$|g(x,u,u_x)| \leq \Gamma,$$

we get in particular that for any grow-up solution only the eigenmodes with $j \leq \sqrt{b}$ can contribute to the limit at infinity. We make this statement precise in the next lemma, derived in [BG11b].

Lemma 2.3.1. *Consider any trajectory $u(t,.)$ of equation (2.1) and the same trajectory in terms of the basis $\{\varphi_j\}_{j\in\mathbb{N}_0}$, which is given by*

$$u(t,x) = \sum_{j=0}^{\infty} \hat{u}_j(t)\varphi_j(x),$$

with $\hat{u}_j(t) = \langle u(t,.), \varphi_j(.)\rangle_{L^2}$. Then all the modes $\hat{u}_j(t)$ with $j > \sqrt{b}$ remain bounded for all time.

Proof. We consider the linear non-homogeneous ODE obtained by taking the projection of equation (2.1) onto the eigenspaces E_j of A associated with λ_j. The obtained equation has the form

$$\frac{d}{dt}\hat{u}_j(t) = -\lambda_j \hat{u}_j(t) + \hat{g}_j(t), \tag{2.4}$$

and we recall that the eigenvalues λ_j for the operator A are given by $j^2 - b$. We get as a general solution for (2.4) the following

$$\hat{u}_j(t) = \hat{u}_j(0)e^{-\lambda_j t} + \int_0^t e^{-\lambda_j(t-s)}\hat{g}_j(s)ds, \tag{2.5}$$

where $\hat{u}_j(0) = \langle u_0(.), \varphi_j(.)\rangle_{L^2}$.

If $\lambda_j > 0$, that is, $j > \sqrt{b}$, then we can get a bound for $\int_0^t e^{-\lambda_j(t-s)}\hat{g}_j(s)ds$, as we can see below:

$$\left|\int_0^t e^{-\lambda_j(t-s)}\hat{g}_j(s)ds\right| \leq \int_0^t e^{-\lambda_j(t-s)}\Gamma ds \leq \frac{\Gamma}{\lambda_j}(1-e^{-\lambda_j t}) \leq \frac{\Gamma}{\lambda_j}.$$

To obtain a bound for the first term $\hat{u}_j(0)e^{-\lambda_j t}$, we just have to notice that for $\lambda_j > 0$

$$|\hat{u}_j(0)e^{-\lambda_j t}| \leq |\hat{u}_j(0)|, \text{ for all } t \geq 0,$$

as long as $\hat{u}_j(0)$ is finite. □

The above result leads us to an important conclusion. If we consider a grow-up solution $u(t,.)$, since its norm grows to infinity with time, the modes $\hat{u}_j$ with $j > \sqrt{b}$ will not affect the shape profile of $u(t,.)$ in the limit. Therefore, once we are interested in the limiting objects at infinity, we are particularly concerned with the modes with $j \leq \sqrt{b}$.

In the next lemma, obtained in [BG11b], it will be considered a grow-up solution and its normalized trajectory. We are interested in a better description of the explicit behavior of grow-up solutions. This will lead us to a crucial result concerning the limit objects of the unbounded solutions, which are the next topic we intend to focus on.

Lemma 2.3.2. *Consider a grow-up solution $u(t,.)$ of (2.1) and its normalized trajectory $\frac{u(t,.)}{\|u(t,.)\|}$. A necessary and sufficient condition for the rescaled trajectory to converge to φ_j^ι in L^2 is that*

$$\lim_{t\to\infty} \frac{\hat{u}_j^2(t)}{\sum\limits_{l=0}^{\infty}\hat{u}_l^2(t)} = 1,$$

and the sign of $\varphi_j^\iota(0)$ should be the same as $u(t,0)$ for all $t \in (T,\infty)$, for some $T > 0$, where $\varphi_j^\iota = \iota\varphi_j$ and $\iota \in \{+1,-1\}$.

Proof. We first calculate the following

$$\begin{aligned}
\|\frac{u(t,.)}{\|u(t,.)\|}-\varphi_j^{\pm}(.)\|^2 &= 1-2\langle\frac{u(t,.)}{\|u(t,.)\|},\pm\varphi_j(.)\rangle+1\\
&= 2\mp\frac{2}{\|u(t,.)\|}\langle u(t,.),\varphi_j(.)\rangle\\
&= 2\mp\frac{2\hat{u}_j(t)}{(\sum\limits_{l=0}^{\infty}\hat{u}_l^2(t))^{\frac{1}{2}}}.
\end{aligned}$$

Then

$$\lim_{t\to\infty}\|\frac{u(t,.)}{\|u(t,.)\|}-\varphi_j^{\pm}(.)\|^2 = 2\mp 2\lim_{t\to\infty}\frac{\hat{u}_j(t)}{(\sum\limits_{l=0}^{\infty}\hat{u}_l^2(t))^{\frac{1}{2}}}. \tag{2.6}$$

We denote by T the largest finite dropping time of the zero number of u. It means that $u(t,0)$ has the same sign for all $T<t<\infty$.

We next prove that the sign of $\varphi_j^{\pm}(0)$ should be the same as $u(t,0)$ for all $t>T$. Suppose that $u(t,0)>0$ for all $t>T$ and $\frac{u(t,.)}{\|u(t,.)\|}$ converges in L^2 to $\varphi_j^{-}(.)$. Since $u(t,.)$ and $\varphi_j^{-}(.)$ are continuous at $x=0$, there exists $\epsilon>0$ such that

$$u(t,x)>0 \text{ and } \varphi_j^{-}(x)<0, \text{ for } x\in[0,\epsilon] \text{ and } t>T.$$

But the L^2-convergence implies that $\epsilon=0$, which contradicts the continuity assumptions. The same argument proves that $\frac{u(t,.)}{\|u(t,.)\|}$ cannot converge in L^2 to $\varphi_j^{+}(.)$ if $u(t,0)<0$ for all $t>T$.

We have learned from the previous lemma that, for any grow-up solution $u(t,.)$, all the modes $\hat{u}_l(t)$ with $l>\sqrt{b}$ must necessarily remain bounded for all time. Therefore, at least one mode $\hat{u}_j$ for some $j<\sqrt{b}$ will grow to infinity with t. Suppose $\hat{u}_j$ is the only infinitely growing mode, i.e., all $\hat{u}_m$ remain bounded if $m\neq j$. If this is the case, then

$$\lim_{t\to\infty}\frac{\hat{u}_j^2(t)}{\sum\limits_{l=0}^{\infty}\hat{u}_l^2(t)}=\lim_{t\to\infty}\frac{\hat{u}_j^2(t)}{\hat{u}_j^2(t)}=1,$$

and

$$\lim_{t\to\infty}\frac{\hat{u}_m(t)}{(\sum\limits_{l=0}^{\infty}\hat{u}_l^2(t))^{\frac{1}{2}}}=\lim_{t\to\infty}\frac{\hat{u}_m(t)}{\hat{u}_j(t)}=0,$$

for all $m\neq j$.

Suppose now that the grow-up solution $u(t,.)$ has more than one infinitely growing mode and denote by $\hat{u}_i$ the one with the lowest subscript. Also suppose that $b\neq l^2$ for any integer l. From equation (2.3) we obtained that, for each integer l

$$\hat{u}_l(t)=\hat{u}_l^h(t)+\hat{u}_l^p(t),$$

where $\hat{u}_l^h(t) = \hat{u}_l^h(0)e^{(b-l^2)t}$, $\hat{u}_l^p(t) = \int\limits_{\infty}^{t} e^{(b-l^2)(t-s)}\hat{g}_l(s)ds$ and

$$\hat{u}_l^h(0) = \hat{u}_l(0) + \int\limits_0^{\infty} e^{(l^2-b)s}\hat{g}_l(s)ds.$$

Moreover, since g is bounded by Γ, we obtain as in the previous lemma that $|\hat{u}_l^p(t)| < \frac{\Gamma}{b-l^2}$. It is then clear that the growth to infinity of $\hat{u}_l(t)$ is determined by the term $\hat{u}_l^h(t)$.

If we then consider any other infinitely growing mode $\hat{u}_j$, we must have $j > i$. From the discussion above on the arbitrary mode $\hat{u}_l$, we conclude that $\hat{u}_i(t)$ grows exponentially faster than $\hat{u}_j$. Therefore,

$$\lim_{t\to\infty} \frac{\hat{u}_j(t)}{\hat{u}_i(t)} = 0$$

and we have the limits

$$\lim_{t\to\infty} \frac{\hat{u}_j(t)}{(\sum\limits_{l=0}^{\infty} \hat{u}_l^2(t))^{\frac{1}{2}}} = \lim_{t\to\infty} \frac{\hat{u}_j(t)}{\hat{u}_i(t)} = 0,$$

and

$$\lim_{t\to\infty} \frac{\hat{u}_i^2(t)}{\sum\limits_{l=0}^{\infty} \hat{u}_l^2(t)} = \lim_{t\to\infty} \frac{\hat{u}_i^2(t)}{\hat{u}_i^2(t)} = 1.$$

Considering instead $\hat{u}_j$ as any bounded mode, the above limits follow immediately.

It remains to consider $b = l^2$ for some integer l. Returning to equation (2.3), it follows that

$$-\Gamma t + \hat{u}_l(0) \leq \hat{u}_l(t) \leq \Gamma t + \hat{u}_l(0),$$

that is, the unbounded mode $\hat{u}_l(t)$ grows at most linearly in t. If $\hat{u}_i$ is an unbounded mode with $i < l$, then we clearly have the limit

$$\lim_{t\to\infty} \frac{\hat{u}_l(t)}{\hat{u}_i(t)} = 0,$$

and then the above calculations apply to all other growing modes.

We conclude that $\frac{u(t,.)}{\|u(t,.)\|}$ will converge to $\varphi_j^{\pm}$ where j denotes the lowest subscript for which $\hat{u}_j$ is an infinitely growing mode, and this is equivalent to $\lim\limits_{t\to\infty} \frac{\hat{u}_j^2(t)}{\sum\limits_{l=0}^{\infty} \hat{u}_l^2(t)} = 1.$ $\square$

It was established in [Hel11] that the projections of the equilibria of the linear equation

$$u_t = u_{xx} + bu$$

to infinite norm play the role of equilibria at infinity. We will next discuss this fact in detail. From the above result we know that the normalized orbit of a grow-up solution converges in L^2 to φ_j^{ι}, for some j. The original orbit corresponding to the solution $u(t,.)$ grows up in the direction of φ_j^{ι}, as the normalized solution converges to φ_j^{ι}.

We take advantage of some recently developed techniques in order to better understand the behavior of $u(t,.)$ at infinity. The essential tool we want to introduce is derived in [Hel11] and consists of a "compactification" of a Hilbert space contained in X, in such a way that it is mapped into an infinite-dimensional manifold. From that we get a huge advantage resulting from the fact that the infinity is mapped onto the boundary of the obtained manifold and we are then able to get a better picture of the behavior at infinity.

We reproduce the discussion in [Hel11] and [BG11c] for the context of our equation (2.1). The objective is to project the Hilbert space

$$X^\alpha := D((A_1)^\alpha),$$

where $\alpha > \frac{3}{4}$, onto the upper hemisphere of an infinite-dimensional sphere so that the infinity is projected onto the equator. We shall next project our equation (2.1) and obtain the Poincaré compactified version of it. The study of the obtained equation will certainly provide us with accurate information on the dynamics at infinity.

The space X^α is a Hilbert space with the inner product given by

$$\langle u, v\rangle_\alpha = \langle (A_1)^\alpha u, (A_1)^\alpha v\rangle_{L^2}$$

(see [Rob01]). Moreover, the inner product $\langle\cdot,\cdot\rangle_\alpha$ induces the norm $\|\cdot\|_\alpha$. We firstly make the natural identification of X^α with the hyperplane $X^\alpha \times \{1\}$ in $X^\alpha \times \mathbb{R}$. The upper hemisphere we intend to project $X^\alpha \times \{1\}$ onto is

$$\mathcal{H} = \{(\chi, z) \in X^\alpha \times \mathbb{R} | \langle \chi, \chi\rangle_\alpha^2 + z^2 = 1, z \geq 0\}.$$

Notice that the hyperplane $X^\alpha \times \{1\}$ is tangent to the unit sphere in $X^\alpha \times \mathbb{R}$. The projection we shall consider is defined in the following way: for a given point M on the hyperplane, the straight line through M and the center of the unit sphere $(0,0)$ intersects the sphere at two antipodal points, one on the upper hemisphere and one in the lower. The projection of M, which will be denoted by $\mathcal{P}(M)$, is defined as the intersection point on the upper hemisphere. In this way, the projection $\mathcal{P}(M)$ has the form (χ, z) with $z \geq 0$. Moreover, as M goes to infinity on $X^\alpha \times \{1\}$, the Poincaré projection of M goes to the boundary of $\mathcal{H}$, that is, the equator of the unit sphere

$$\mathcal{H}_e := \{(\chi, 0) \in X^\alpha \times \mathbb{R} | \langle \chi, \chi\rangle_\alpha = 1\}.$$

We refer to $\mathcal{H}_e$ as the "sphere at infinity".

We are actually able to give an explicit formula for the projection $\mathcal{P}(M)$, since the center of unit sphere, the point M and $\mathcal{P}(M)$ are all collinear. For $M = (u,1) \in X^\alpha \times \{1\}$, the Poincaré projection of M is given by

$$\mathcal{P}(M) = (\chi, z) = \left(\frac{u}{(1 + \langle u, u\rangle_\alpha)^{1/2}}, \frac{1}{(1 + \langle u, u\rangle_\alpha)^{1/2}}\right). \tag{2.7}$$

We next project M onto several tangent hyperplanes of $X^\alpha \times \mathbb{R}$.

The Hilbert space X^α is provided with the countable orthonormal basis $\{\varphi_n^\alpha\}_{n\in\mathbb{N}_0}$ with $\varphi_n^\alpha := A_1^{-\alpha}\varphi_n$. The next step is to fix a basis vector $\pm\varphi_j^\alpha$ and project $\mathcal{P}(M)$ onto the vertical hyperplane C_j which is tangent to the equator $\mathcal{H}_e$ at $(\pm\varphi_j^\alpha, 0)$. The projection of $\mathcal{P}(M)$ will be defined as the point in C_j which is colinear with M and $\mathcal{P}(M)$. The sufficient condition for such a projection to be well-defined is that the line through $\mathcal{P}(M)$, M and the origin of the unit sphere intersects C_j. If we denote the projected point by $M' = (\xi, \zeta) \in C_j$ the projection is given explicitly by

$$(\xi, \zeta) = \frac{1}{\langle u, \pm\varphi_j^\alpha \rangle_\alpha}(u, 1) = \frac{1}{\langle \chi, \pm\varphi_j^\alpha \rangle_\alpha}(\chi, z) \tag{2.8}$$

and the condition for the projection to be well-defined is

$$\langle u, \pm\varphi_j^\alpha \rangle_\alpha > 0 \text{ or equivalently } \langle \chi, \pm\varphi_j^\alpha \rangle_\alpha > 0.$$

The hyperplane corresponding to $\pm\varphi_j^\alpha$ is

$$\{(\xi, \zeta) \in X^\alpha \times \mathbb{R} | \xi_j = \langle \xi, \varphi_j^\alpha \rangle_\alpha = \pm 1\}$$

and (2.8) is then equivalently given by

$$\xi_n = \pm\frac{\hat{u}_n}{\hat{u}_j}, \text{ for all } n \in \mathbb{N}, \quad \zeta = \pm\frac{1}{\hat{u}_j} \tag{2.9}$$

for all $u \in X^\alpha$ with jth coordinate nonzero. As pointed out by [BG10], the collection of projections defined in this way for each $j \in \mathbb{N}_0$ builds an atlas of $\mathcal{H} \setminus \{(0, 1)\}$.

After describing the geometric aspects of the Poincaré projection, we now pass to our differential equation and study how an equation of the form

$$u_t = \mathcal{N}(u) = u_{xx} + bu + g(x, u, u_x) \tag{2.10}$$

on X^α is transformed by the projection $\mathcal{P}$. The objective is to examine the obtained equation on the equator. We begin by recalling that $\lambda_j = j^2 - b$ is the jth eigenvalue of the operator $A = -\partial_{xx} - bI$. Furthermore, we use the notation

$$\mathcal{N}_\zeta := \zeta\mathcal{N}(\zeta^{-1}), \quad g_\zeta(x, u, u_x) := \zeta g(x, \zeta^{-1}u, \zeta^{-1}u_x),$$

where $\mathcal{N}_\zeta$ denotes the homothety of $\mathcal{N}$ with factor $\zeta \neq 0$. Taking the derivatives of (2.8) with respect to time, it leads to the following:

$$\xi_t = \mathcal{N}(u)\zeta \mp u\zeta\langle \mathcal{N}(u)\zeta, \varphi_j^\alpha \rangle_\alpha = \mathcal{N}_\zeta(\xi) \mp \langle \mathcal{N}_\zeta(\xi), \varphi_j^\alpha \rangle_\alpha \xi$$

$$\zeta_t = \mp\langle \mathcal{N}(u)\zeta, \varphi_j^\alpha \rangle_\alpha \zeta = \mp\langle \mathcal{N}_\zeta(\xi), \varphi_j^\alpha \rangle_\alpha \zeta.$$

By writing the above equations in spectral projection coordinates, $\xi_n = \langle \xi, \varphi_n^\alpha \rangle_\alpha$, and taking advantage of the eigenvalues λ_j we get, for each j, the equations below in the half-hyperplanes

$$\{(\xi,\zeta)\in X^1\times\mathbb{R}|\xi_j=\langle\xi,\varphi_j^\alpha\rangle_\alpha=\pm1\ \ and\ \ \zeta\geq 0\}\subset C_j:$$

$$(\xi_n)_t=\pm(j^2-n^2)\xi_n\mp\langle g_\zeta(x,\xi,\xi_x),\varphi_j^\alpha\rangle_\alpha\xi_n+\langle g_\zeta(x,\xi,\xi_x),\varphi_n^\alpha\rangle_\alpha \tag{2.11}$$

$$\zeta_t=\pm(j^2-b)\zeta\mp\langle g_\zeta(x,\xi,\xi_x),\varphi_j^\alpha\rangle_\alpha\zeta.$$

In the following we want to obtain the projected version of the above equations on the equator of the unit sphere. For that we aim to examine the equations as ζ goes to zero. Notice that g_ζ and $\langle g_\zeta(x,\xi,\xi_x),\varphi_j^\alpha\rangle_\alpha$ converge to zero as ζ converges to zero, from the definition of g_ζ and from the bound we imposed on g. Then, we get in particular that the equator is in fact invariant, since ζ_t converges to zero as $\zeta\to 0$. Therefore, (2.11) on the equator has the form

$$(\xi_n)_t=(j^2-n^2)\xi_n, \tag{2.12}$$

for all $j\neq n$. We are finally able to get the equilibrium points on the equator. They are given on C_j by

$$\{(\xi,\zeta)|\xi_j=\pm1,\ \zeta=0\ \text{ and } \xi_n=0\ \forall n\neq j\}.$$

Or equivalently, with coordinates in the Poincaré hemisphere $\mathcal{H}$, we have

$$\Phi_j^\pm=\{(\chi,z)|\chi_j=\pm1,\ z=0\ \text{ and } \chi_n=0\ \forall n\neq j\}. \tag{2.13}$$

Because the infinity of X^α is projected onto the equator $\mathcal{H}_e$ and $\Phi_j^\pm$ are equilibrium points on $\mathcal{H}_e$, we define objects $\Phi_j^{\infty,\pm}$ at infinity as

$$\mathcal{P}(\Phi_j^{\infty,\pm})=\Phi_j^\pm,$$

and refer to these as equilibria at infinity. We also say that the equilibrium at infinity $\Phi_j^{\infty,\pm}$ is hyperbolic if the corresponding equilibria at the equator $\Phi_j^\pm$ is hyperbolic. It is then clear that the restriction $b\neq n^2$ ensures hyperbolicity for all the equilibria at infinity.

As we have noticed when considering the Poincaré projection, the equation $u_t=u_{xx}+bu+g(x,u,u_x)$ is transformed in such a way that in spectral projection coordinates it becomes (2.11). The obtained system, on its turn, converges to (2.12) as $\zeta\to 0$, i.e., as the norm of the corresponding function u grows to infinity. On the other hand, the linear equation $u_t=u_{xx}+bu$ with Neumann boundary conditions has lines of equilibria through the vectors $\{\pm\varphi_j\}_{j\in\mathbb{N}}$. Then, the equilibria at infinity of the linear equation are projected onto $\{\Phi_j^\pm\}_{j\in\mathbb{N}}$. In this way, given a grow-up solution, we know that its normalized orbit converges in L^2 to some equilibria $\varphi_j^\pm$. Then the original orbit grows up to infinity in the direction of $\varphi_j^\pm$ and its Poincaré projection converges to $\Phi_j^\pm$.

We can certainly affirm now that the objects $\Phi_j^{\infty,\pm}$ play the role of equilibria at infinity for the equation (2.1). We are even able to get a better understanding of such objects, as well as to examine the existence of orbits connecting them. But before we pass to a detailed discussion concerning connections between both bounded and unbounded equilibria, we introduce in the next section some geometric methods which have already proved to be of paramount relevance regarding the analysis of parabolic PDE's (see, for instance, [Ang88, BF88, Mat82]).

Chapter 3

Nodal properties and Y-map

We utilize nodal properties functionals, as the zero number and the intersection number, to study the heteroclinic connections between the equilibria. These functionals of solutions for equation (2.1) possess important properties which are extremely useful in significantly restricting the possible limits for any trajectory. Another relevant functional, firstly introduced in [BF88] for dissipative semilinear parabolic PDEs, is called y-map and it provides information on the nodal properties of solutions converging in backwards to an equilibrium. In this section we shall define and discuss some properties of such functionals.

Given a continuous function $u(x)$ defined on an interval I, the zero number of u, $z(u)$, denotes the number of strict sign changes of $x \mapsto u(x)$. We set the zero number of a constant function to be zero. If u is a function of $t \in \mathbb{R}$ and $x \in I$, then the zero number is defined for each $t \in \mathbb{R}$ as $z(u(t,.))$. Moreover, if u and v are continuous functions defined on an interval I, we define the intersection number of the two functions as the zero number of their difference, $z(u - v)$.

As X^α embeds into C^1 for $\alpha > \frac{3}{4}$, the zero number z is well defined on X^α. We thus have the following result from [Ang88].

Proposition 3.0.1. *Let $\tilde{u}(t,.) \in X^\alpha$ be a non-trivial solution of the linear equation*

$$\tilde{u}_t = \tilde{u}_{xx} + c(t,x)\tilde{u}_x + d(t,x)\tilde{u},$$

with Neumann boundary conditions at $x = 0, \pi$. Assume c and d are continuously differentiable. Then
(i) $z(\tilde{u}(t,.))$ is finite for any $t > 0$,
(ii) if (t_0, x_0), for some $t_0 > 0$ and $0 \leq x_0 \leq \pi$, is a multiple zero of u, i.e., $\tilde{u}(t_0, x_0) = \tilde{u}_x(t_0, x_0) = 0$, then $z(\tilde{u}(t_1,.)) > z(\tilde{u}(t_2,.))$ for all $t_1 < t_0 < t_2$.

Although the y-map was firstly introduced for dissipative semilinear parabolic equations with Dirichlet or mixed boundary conditions, in [BG11b] an extended form of the y-map was designed to deal with a class of non dissipative equations with Neumann boundary conditions. Throughout this section, we reproduce the discussion in [BG11b] concerning the y-map with a slight adjustment to include the case where the nonlinearity also depends on u_x.

Consider the equation

$$\begin{cases} u_t = u_{xx} + bu + g(x,u,u_x), & x \in [0,\pi] \\ u_x(t,0) = u_x(t,\pi) = 0 \end{cases}, \tag{3.1}$$

where $g \in \mathcal{G}$ and

$$\mathcal{G} = \left\{ g(x,u,u_x) \in C^2 | g(x,u,u_x) \text{ is bounded uniformly} \right\}. \tag{3.2}$$

We also denote by

$$\mathcal{F} = \{ f(x,u,u_x) | f(x,u,u_x) = bu + g(x,u,u_x) \text{ and } g \in \mathcal{G} \}. \tag{3.3}$$

Moreover, $\mathcal{F}$ and $\mathcal{G}$ are both endowed with the weak Whitney topology, [Hir76].

Before we introduce the y-map, we shall first restrict $\mathcal{F}$ and $\mathcal{G}$. We impose on $f \in \mathcal{F}$ and $g \in \mathcal{G}$ the following

$$f(x,0,0) = g(x,0,0) = 0, \quad for \;\; all \;\; x \in [0,\pi]$$

and denote by $\mathcal{F}_0$ and $\mathcal{G}_0$ the obtained restricted sets. We further notice that, with such a restriction, the zero number of any solution of equation (3.1) satisfies the crucial property of being nonincreasing as described in Proposition 3.0.1.

Timely to remark that any equation in the form (3.1) with bounded equilibria can be written in such a way that the obtained equation in $\tilde{u}$ possesses nonlinearities $\tilde{f}$ and $\tilde{g}$ in $\mathcal{F}_0$ and $\mathcal{G}_0$, respectively. In fact, given any bounded equilibrium v of equation (3.1), if we introduce the change of variables

$$\tilde{u} = u - v,$$

then equation (3.1) can be rewritten as

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + b\tilde{u} + \tilde{g}(x,\tilde{u},\tilde{u}_x), & x \in [0,\pi] \\ \tilde{u}_x(t,0) = \tilde{u}_x(t,\pi) = 0 \end{cases}, \tag{3.4}$$

where $\tilde{g}(x,\tilde{u},\tilde{u}_x) = g(x, v+\tilde{u}, v_x+\tilde{u}_x) - g(x,v,v_x) \in \mathcal{G}_0$.

In what follows we define the y-map assuming that an eventually necessary change of variables was already made and we had replaced $\tilde{u}$ with u. The y-map is constructed as a continuous mapping

$$y : \{u_0 \in X^\alpha | z(u_0) \leq n, u_0 \neq 0\} \to S^n \subset \mathbb{R}^{n+1},$$

where S^n denotes the n-dimensional sphere.

Let $u_0 \in X^\alpha \setminus \{0\}$ such that $z(u_0) \leq n$ and let $u(t,.)$ denote the corresponding trajectory. We define the dropping times $t_k \in [0,\infty]$ as the first time that the zero number $z(u(t,.))$ drops to or below the value

k:

$$t_k := \inf\{t \geq 0 | z(u(t,.)) \leq k\}.$$

Since the zero number $z(u(t,.))$ is nonincreasing with t, then

$$0 = t_n \leq t_{n-1} \leq ... \leq t_0.$$

Also notice that, if we define $\tau_k := \tanh(t_k) \in [0,1]$ then

$$0 = \tau_n \leq \tau_{n-1} \leq ... \leq \tau_0.$$

We further define the sign of each element of the y-map by

$$\iota_k := \begin{cases} \operatorname{sign} u(t,0) \text{ for some } t \in (t_k, t_{k-1}), & if\ t_k < t_{k-1} \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

The coordinates $y_1, ..., y_n$ of the y-map are defined by

$$y_0 := \iota_0(1-\tau_0)^{\frac{1}{2}}$$

$$y_k := \iota_k(\tau_{k-1} - \tau_k)^{\frac{1}{2}}, 1 \leq k \leq n.$$

It follows that the ι_k are well-defined by the next lemma, derived in [BG11b], which states in particular that $u(t,0) \neq 0$ for $t \in (t_k, t_{k+1})$. At this point, we should notice that y in fact maps to S^n.

Lemma 3.0.3. *Suppose $f \in \mathcal{F}_0$ and $z(u(0,.)) < \infty$. Then the set of times $t > 0$ such that $x \mapsto u(t,x)$ has only simple zeros is open and dense in $\mathbb{R}^+$. Further, if we define the dropping times as above and $t_k < t_{k-1}$, then*

$$u(t,0) \neq 0 \ \ \forall t \in (t_k, t_{k-1}).$$

Proof. Throughout this proof we use the previous notation wherein $u(t,x)$ solves equation (3.1) and $\tilde{u}(t,x)$ is a solution of the shifted equation (3.4). We can rewrite the shifted equation (3.4) in the form

$$\tilde{u}_t = \tilde{u}_{xx} + c(t,x)\tilde{u}_x + d(t,x)\tilde{u}, \tag{3.6}$$

where

$$c(t,x) = \int_0^1 g_p(x, \theta u + (1-\theta)v, \theta u_x + (1-\theta)v_x) d\theta$$

and

$$d(t,x) = b + \int_0^1 g_u(x, \theta u + (1-\theta)v, \theta u_x + (1-\theta)v_x) d\theta,$$

with g_u and g_p denoting the partial derivative of $g(x,u,p)$ with respect to second and third variable respectively.

By Proposition 3.0.1, if $u(t,.)$ is a solution of (3.6) and $u(t_0,x_0) = u_x(t_0,x_0) = 0$ for some $t_0 > 0$ and $x_0 \in [0,\pi]$, then t_0 is a dropping time. If $x_0 = 0$, then for all $t \in (t_k, t_{k-1})$ with $t_k \neq t_{k-1}$ we must have $u(t,0) \neq 0$. In fact, if that was not the case then by the Neumann boundary conditions $u(t,.)$ would have $x_0 = 0$ as a multiple zero and then t would be a dropping time, which is a contradiction since $t_k < t < t_{k-1}$. Moreover, notice that $z(\tilde{u}(t,.))$ being finite implies that the number of dropping times is also finite. Therefore, the set of times $t > 0$ such that $x \mapsto u(t,x)$ has only simple zeros is open and dense in $\mathbb{R}^+$. $\square$

The y-map provides crucial information on the zero number of solutions in the unstable manifold of equilibria. It is then important to recover from $y(u_0)$, for any given u_0, the dropping times t_k and signs ι_k. When, for instance, $y(u_0) = \iota e_k$, where e_k denotes the k-th unit vector and $\iota \in \{1,-1\}$, we shall have $0 = t_n = ... = t_k$ and $\infty = t_{k-1} = ... = t_0$. From the definition of the y-map, it implies that

$$z(u_0) = k, \;\; z(u(t,.)) = k \;\; \forall 0 < t < \infty.$$

Moreover, $\iota_i = 0$ for all $i \in \{1,...,n\} \setminus \{k\}$ and $\iota_k = \operatorname{sign} u(t,0)$ for some $t \in (0,\infty)$, but we know from Proposition 3.0.1 that the sign of $u(t,0)$ cannot change for any non-dropping time and hence

$$\iota_k \cdot u(t,0) > 0$$

for all $0 < t < \infty$. With this in mind, our next step is to prove the surjectivity of the y-map.

As one can easily notice, it is significantly simpler to consider f linear when approaching the surjectivity of y. Then one can use a homotopy deforming f from linear to nonlinear. It is appropriate to mention that the y-map has continuous dependence on $f \in \mathcal{F}_0$ and $u_0 \in X^\alpha \setminus \{0\}$ with $z(u_0) \leq n$. The proof of this result in the slowly non-dissipative case and under Neumann boundary conditions is in [BG10]. A continuous map is said to be essential if it is not homotopic to a constant map. As a result, if the y-map

$$y : \Sigma^n \longrightarrow S^n,$$

restricted to the n-dimensional sphere Σ^n in the unstable manifold of the trivial solution, is an essential mapping then it must be surjective. In fact, if the image of y skips one point in S^n, it will imply a contraction to a single point which will lead to an homotopy from y to the constant map, contradicting the fact that y is essential. Also notice that, y being essential implies that y remains surjective under an homotopy, since the property of being essential is invariant under homotopies to nonlinear f.

Firstly, we take f linear, i.e., $f(x,u,u_x) = b(x)u$ where $b \in C^2$. We consider the Sturm-Liouville problem associated to equation (3.1)

$$\begin{cases} u_{xx} + b(x)u = \lambda u \\ u_x(0) = u_x(\pi) = 0 \end{cases} \tag{3.7}$$

and we denote by $\lambda_0 > \lambda_1 > ...$ and $\varphi_0, \varphi_1, ...$ the corresponding eigenvalues and eigenfunctions. We assume that the $\varphi_i(x)$ are renormalized to unit length in the X-norm and we also choose the sign convention $\varphi_i(0) > 0$. Assume that $\lambda_n > 0$, i.e., $u \equiv 0$ has Morse index $i(u \equiv 0) \geq n+1$. Denoting

$$W_n = \operatorname{span}\{\varphi_0, ..., \varphi_n\},$$

it follows from Sturm Liouville theory that $z(w) \leq n$ for $w \in W_n$, [BR78].

Taking into account the above notation and assumption, we have the following result for f linear (see [BF88] and [BG11b]).

Lemma 3.0.4. *The y-map restricted to Σ^n possesses the property of being essential and, in particular surjective.*

The following result, established in [BF88] and [BG11b], and restated here, proves the surjectivity of the y-map for general nonlinear f. We obtained Lemma 3.0.4 for the linear case and the next result for the nonlinear case is obtained by standard homotopy.

Lemma 3.0.5. *Consider equation (3.1) with $g \in \mathcal{G}_0$. Let v be a hyperbolic equilibrium and let W^u denote the unstable manifold of v with dimension $i(v) = n+1 > 0$. Let $\Sigma \subset W^u \setminus \{v\}$ be homotopic in $W^u \setminus \{v\}$ to a small sphere in W^u centered in at v of dimension n. Then for any finite sequence*

$$0 = \delta_n \leq \delta_{n-1} \leq ... \leq \delta_0 \leq \infty$$

$$s_k \in \{1, -1\},\ 0 \leq k \leq n,$$

there exists a point $u_0 \in \Sigma$ corresponding to an initial condition $u(0,.) \in X^\alpha$ such that the graph $t \mapsto z(u(t,.) - v(.))$ is determined by (δ_k). More precisely, for any $0 \leq t < \infty$,

$$t \geq \delta_k \Leftrightarrow z(u(t,.) - v(.)) \leq k$$

$$\delta_k < t < \delta_{k-1} \Rightarrow \operatorname{sign}(u(t,0) - v(0)) = s_k.$$

Proof. We first prove that the restricted y-map is essential. We then homotopically deform f from the corresponding linear form, defining

$$f_\vartheta(x, u, u_x) := bu + g_\vartheta(x, u, u_x) := bu + \vartheta g(x, u, u_x) +$$

$$+ (1-\vartheta)[g_u(x, 0, 0) \cdot u + g_p(x, 0, 0) \cdot u_x]$$

with homotopy parameter $0 \leq \vartheta \leq 1$. As we deform f, the unstable manifold of the equilibrium $v \equiv 0$ of (3.1) with a specific nonlinearity f_ϑ is simultaneously deformed. Also notice that the linearization at $v \equiv 0$ in the homotopically deformed system is unchanged, as we can see in the following

$$0 = u_{xx} + bu + \vartheta g_u(x, 0, 0)u + (1-\vartheta)g_u(x, 0, 0)u +$$

$$+\vartheta g_p(x,0,0)u_x + (1-\vartheta) g_p(x,0,0)u_x$$

$$= u_{xx} + bu + g_u(x,0,0)u + g_p(x,0,0)u_x.$$

Moreover, $f_\vartheta \in \mathcal{F}_0$ depends continuously on ϑ since $\mathcal{F}_0$ carries the weak Whitney topology.

Let

$$W^u_{loc}(f_0) := \mathrm{span}\{\varphi_0, ..., \varphi_n\} \cap \{u_0 \in X| \;\; |u_0| < 2\epsilon\}$$

denote the cut-off tangent space $W^u(f_\vartheta)$ at $v \equiv 0$ for $\vartheta = 0$. Then the local unstable manifolds with respect to an f_ϑ are parametrized by diffeomorphisms

$$\rho_\vartheta : W^u_{loc}(f_0) \longrightarrow W^u_{loc}(f_\vartheta)$$

where ρ_ϑ^{-1} is induced by the orthogonal projection onto $\mathrm{span}\{\varphi_0, ..., \varphi_n\}$. Notice that ρ_ϑ depends continuously on ϑ in the uniform C^0 topology.

Fix a sphere

$$\Sigma^n := \{u \in W^u_{loc}(f_0)| \;\; |u| = \epsilon\}$$

in $W^u_{loc}(f_0)$ and let y^ϑ denote the restriction to $\rho_\vartheta(\Sigma^n)$ of the y-map associated to f_ϑ. We may assume $\Sigma = \rho_1(\Sigma^n)$, after a homotopy. We thus can define

$$\overline{y}_\vartheta := y^\vartheta \cdot \rho_\vartheta : \Sigma^n \longrightarrow S^n,$$

since $z(u) \leq n$ on $W^u(f_\vartheta)$, [BF86]. This mapping is continuous and depends continuously on ϑ due to its continuous dependence on f and u_0. Lemma 3.0.4 implies that

$$\overline{y}_0 = y^0 \cdot \rho_0 = y^0 : \Sigma^n \longrightarrow S^n$$

is essential. By homotopy invariance of this property, $\overline{y}_1 = y^1 \cdot \rho_1 = y \cdot \rho_1$ is essential and hence y is essential.

The restricted y-map

$$y : \Sigma \longrightarrow S^n$$

being essential implies that it is surjective. Then define the vector ς just as the y-map was defined, but replacing t_k with δ_k and ι_k with s_k. By the surjectivity of y, there exists an initial data $u_0 \in \Sigma$ such that $y(u_0) = \varsigma$. But as we noticed before, $y(u_0)$ uniquely determines the dropping times t_k and signs ι_k of the solution $u(t,.)$ corresponding to u_0. Thus, it is determined that $t_k = \delta_k$ and $\iota_k = s_k$ whenever $\delta_k < \delta_{k-1}$. $\square$

Chapter 4

Heteroclinics to infinity and the bounded solutions

The aim of this and the next chapter is to give a decomposition of the non-compact global attractor $\mathcal{A}_f$. We will first draw our attention to the grow-up solutions, since these trajectories consist of heteroclinic connections to infinity, which will be defined as a connection from a bounded to an unbounded equilibrium. We will then present a brief discussion on the heteroclinics at infinity and conclude with a discussion on the set of bounded solutions of equation (2.1). However an exact criterion to solve the problem of heteroclinic connections among the bounded equilibria will be presented only in the next chapter.

4.1 Asymptotics of grow-up solutions

As we have noticed before, the non-compact global attractor is comprised of a bounded and an unbounded subset. The former, henceforth denoted by $\mathcal{A}_f^c$, is composed of the set of solutions in $\mathcal{A}_f$ which remain bounded in the state space X^α for $t \geq 0$. Given that, we may apply the standard theory for dissipative equations and conclude that the bounded subset $\mathcal{A}_f^c$ is entirely composed of bounded equilibria and orbits connecting them (see [Hal88]).

We can then affirm that the global attractor is defined as

$$\mathcal{A}_f = \mathcal{A}_f^c \cup \mathcal{A}_f^\infty,$$

where $\mathcal{A}_f^c$ is a compact set contained in some sufficiently large ball $B \subset X^\alpha$, and $\mathcal{A}_f^\infty$ is the unbounded part of $\mathcal{A}_f$ which will be next described.

We turn our attention to the unbounded subset $\mathcal{A}_f^\infty$ of the non-compact global attractor. This means that the grow-up solutions of (2.1) are now the center of our investigation. We have learned previously that the solutions of equation (2.1) whose norm grows to infinity with time converge to some objects at infinity, and these objects consist of infinite projections of the eigenfunctions of the Laplacian with

Neumann boundary conditions.

Later we intend to determine the bounded equilibria connecting to infinity as well as the exact limit of the grow-up solutions. In order to do that, it would be crucial to be able to work in higher norms that the L^2-norm. This is due to the fact that the L^2-norm is insufficient for determining the limiting object at infinity and the y-map alone does not prohibit the zero number to drop at ∞. As it was observed in [BG10], despite the fact that the influence of the bounded nonlinearity g must decrease as the norm of $u(t, .)$ grows to infinity, we are unable to study the limit in the X^α norm.

To obtain the limits in C^1, we appeal to the theory of inertial manifolds. For slowly non-dissipative equations in the form (2.1) with $g(x, u, u_x) \equiv g(u)$, it was proved in [BG11a] the existence of a finite dimensional, unbounded, exponentially attracting and positively invariant manifold, i.e., an inertial manifold, provided a spectral gap condition is satisfied. In the setting of equation (2.1), this means that there exists an inertial manifold $\mathcal{M}$ which is the graph of a function

$$\Psi : P_N X^\alpha \longrightarrow Q_N X^\alpha$$

which is Lipschitz with values in $X^\alpha \subset C^1$, with $\alpha > \frac{3}{4}$ and N sufficiently large so that

$$N > \max\{\frac{4L+1}{2}, \sqrt{b}\},$$

where L denotes the Lipschitz coefficient of the Nemitskii operator G of g,

$$G : X^\alpha \longrightarrow X,$$

$P_N : X \to X$ is the orthogonal projection onto $\{\varphi_0, \varphi_0, ..., \varphi_N\}$ and $Q_N = I - P_N$ is the projection onto the orthogonal complement of $P_N X$ (see [BG11b]).

However, if the nonlinearity g in fact exhibits dependence on u_x, we obtain a spectral gap condition implying in a more restrictive sufficient condition for the existence of $\mathcal{M}$. This will be next stated precisely and the result follows from the following lemma, obtained in [Mik91] and presented here in terms of our setting. It is worth noticing that we want to consider nonlinearities in the form $g(x, u, u_x)$ which are globally Lipschitz in the second and third variables, that is, that there exist constants L_1 and L_2 such that

$$|g(x, u^1, p^1) - g(x, u^2, p^2)| \leq L_1|u^1 - u^2| + L_2|p^1 - p^2|.$$

Lemma 4.1.1. *Under the following conditions:*
(i) $G : X^\alpha \longrightarrow X$ *is continuous and for some bounded operators* $B_1, B_2 : X^\alpha \longrightarrow X$ *we have that*

$$\|G(u^1) - G(u^2)\| \leq \|B_1(u^1 - u^2)\| + \|B_2(u^1 - u^2)\|,$$

for all $u^1, u^2 \in X^\alpha$ *and* $\alpha \in [0, 1)$,
(ii) *for some* $\lambda \in \mathbb{R}$, *with* $\lambda > 0$, $\lambda + i\omega$ *is in the resolvent of* A *for all* $\omega \in \mathbb{R}$, (iii) *for* λ *as before, we have*

that

$$\sup_{\omega\in\mathbb{R}} \|B_1(A-\lambda-i\omega)^{-1}\| + \sup_{\omega\in\mathbb{R}} \|B_2(A-\lambda-i\omega)^{-1}\| < 1,$$

there exists a N-dimensional inertial manifold $\mathcal{M}$ for equation (2.1). Moreover, $\mathcal{M}$ is the graph of a Lipschitz function from X to X^α.

We thus have the following result where the conditions on the nonlinearity g are specified.

Lemma 4.1.2. *Consider equation (2.1) with g satisfying the inequality*

$$|g(x,u^1,p^1) - g(x,u^2,p^2)| \le L_1|u^1-u^2| + L_2|p^1-p^2| \tag{4.1}$$

for some $L_1 > 0$ and for $L_2 < 1$. Then there exists an inertial manifold $\mathcal{M}$ for equation (2.1) and $\mathcal{M}$ is Lipschitz in C^1.

Proof. We first recall that (2.1) can be written in the form

$$u_t = Au + G(u),$$

where $A = \partial_{xx} + bI$ and G is the Nemitskii operator of g. Notice that the operators

$$B_1 u := L_1 u \text{ and } B_2 u := L_2 u_x$$

are bounded from X^α into X and, if $u, v \in X^\alpha$, then

$$\|G(u^1) - G(u^2)\| \le \|B_1(u^1-u^2)\| + \|B_2(u^1-u^2)\|,$$

by (4.1). Then condition (i) in the previous lemma is satisfied. Moreover, the eigenvalues of A with Neumann boundary conditions are given by $\lambda_j = j^2 - b,\ \ j \in \{0,1,...\}$ and, if we fix $N \in \mathbb{N}$, then $\lambda + i\omega$ is in the resolvent of A for all $\omega \in \mathbb{R}$ if $\lambda \neq \lambda_j$. In particular, we can consider

$$\lambda := \frac{\lambda_N + \lambda_{N+1}}{2}.$$

Condition (ii) in Lemma 4.1.1 is then verified.

We shall next calculate bounds for

$$S_1 := \sup_{\omega\in\mathbb{R}} \|B_1(A-\lambda-i\omega)^{-1}\|$$

and

$$S_2 := \sup_{\omega\in\mathbb{R}} \|B_2(A-\lambda-i\omega)^{-1}\|.$$

The operator $(A-\lambda-i\omega)^{-1}$ has the following spectral representation

$$\sum_{j=0}^{\infty} \frac{1}{\lambda_j - \lambda - i\omega} E_j,$$

where E_j is the projection on the jth eigenfunction, i.e.,

$$E_j(u)(x) = \langle u, \varphi_j \rangle_{L^2} \varphi_j(x).$$

We thus have

$$\begin{aligned}
S_1 &= L_1 \sup_{\omega \in \mathbb{R}} \|(A - \lambda - i\omega)^{-1}\| \\
&= L_1 \sup_{\omega \in \mathbb{R}} \| \sum_{j=0}^{\infty} \frac{1}{\lambda_j - \lambda - i\omega} E_j \| \\
&= L_1 \| \sum_{j=0}^{\infty} \frac{1}{\lambda_j - \lambda} E_j \| \\
&\leq L_1 \max\{ | \frac{1}{\lambda_N - \lambda} |, | \frac{1}{\lambda_{N+1} - \lambda} | \} \| \sum_{j=0}^{\infty} E_j \| \\
&\leq L_1 \max\{ | \frac{1}{\lambda_N - \lambda} |, | \frac{1}{\lambda_{N+1} - \lambda} | \} \\
&= \frac{L_1}{N + 1/2}.
\end{aligned}$$

We similarly obtain for S_2 the following

$$\begin{aligned}
S_2 &= L_2 \sup_{\omega \in \mathbb{R}} \|\partial_x (A - \lambda - i\omega)^{-1}\| \\
&\leq L_2 \sup_{\omega \in \mathbb{R}} \|(A + b)^{1/2} (A - \lambda - i\omega)^{-1}\| \\
&= L_2 \sup_{\omega \in \mathbb{R}} \| \sum_{j=0}^{\infty} \frac{(\lambda_j + b)^{1/2}}{\lambda_j - \lambda - i\omega} E_j \| \\
&= L_2 \| \sum_{j=0}^{\infty} \frac{j}{\lambda_j - \lambda} E_j \| \\
&\leq L_2 \max\{ | \frac{N}{\lambda_N - \lambda} |, | \frac{N+1}{\lambda_{N+1} - \lambda} | \} \| \sum_{j=0}^{\infty} E_j \| \\
&\leq L_2 \max\{ | \frac{N}{\lambda_N - \lambda} |, | \frac{N+1}{\lambda_{N+1} - \lambda} | \} \\
&= L_2 \frac{N+1}{N + 1/2}.
\end{aligned}$$

We thus conclude that the condition (iii) in Lemma 4.1.1 is satisfied as long as

$$\frac{L_1}{N + 1/2} + L_2 \frac{N+1}{N + 1/2} < 1.$$

Therefore, if $L_2 < 1$ and N is sufficiently large then we are guaranteed the existence of a N-dimensional inertial manifold for equation (2.1). $\square$

The above result entails a restriction on the nonlinearity g that we must consider from now on, as we will strongly rely on the existence of $\mathcal{M}$. We will henceforth consider the equation

$$u_t = u_{xx} + bu + g(x, u, u_x), \quad x \in [0, \pi] \tag{4.2}$$
$$u_x(t, 0) = u_x(t, \pi) = 0$$
$$b > 0, \quad g \in C^2, \quad g \text{ uniformly bounded,}$$

and $g : \mathbb{R}^3 \to \mathbb{R}$ also satisfies

$$|g(x, u^1, p^1) - g(x, u^2, p^2)| \leq L_1|u^1 - u^2| + L_2|p^1 - p^2|,$$

for some $L_1 > 0$ and for $L_2 < 1$.

In view of $\mathcal{M}$ being forward invariant and exponentially attracting, it follows that it contains all invariant sets, including all the grow-up solutions whose limits we aim to establish. In particular, $\mathcal{A}_f$ cannot be bounded since it must contain at least one unbounded solution. Moreover, the non-compact global attractor must be contained in the obtained non-compact inertial manifold.

The next lemma determines the limiting object at infinity by just requiring information on the zero number of the grow-up solution. It essentially states that a grow-up trajectory with shifted zero number k for $0 \leq t < \infty$ converges to an object at infinity with zero number k.

Lemma 4.1.3. *Let v be a hyperbolic equilibrium for equation (4.2) and $u(t, .)$ a grow-up solution in the unstable manifold of v, $W^u(v)$, with*

$$z(u(t, .) - v(.)) = k$$

$$\iota = \mathrm{sign}(u(t, 0) - v(0)),$$

for $0 \leq t < \infty$. Then

$$\lim_{t \to \infty} \| \frac{u(t, .)}{\|u(t, .)\|} - \varphi_k^\iota \|_{C^1} = 0,$$

where $\varphi_k^\iota = \iota \varphi_k$ and $\iota \in \{-1, +1\}$.

Proof. Recalling the existence of an inertial manifold $\mathcal{M} = \mathrm{graph}[\Psi]$ where

$$\Psi : P_N D(A) \longrightarrow Q_N D(A)$$

is a C^1 Lipschitz mapping, we may decompose $u(t, .)$ as follows:

$$u(t, .) = p(t, .) + q(t, .)$$

where

$$p(t, .) = \sum_{i=0}^{N} \langle u(t, .), \varphi(.) \rangle_{L^2} \in P_N D(A)$$

$$q(t, .) = \Psi(p(t, .)) \in Q_N D(A).$$

We know from Lemma 2.3.1 that all modes $\hat{u}_j(t)$ with $j > \sqrt{b}$ remain bounded and also, from Lemma

2.3.2, it follows that

$$\lim_{t\to\infty} \|\frac{u(t,.)}{\|u(t,.)\|} - \varphi_i^t\| = 0$$

for some integer i. Then $i < \sqrt{b}$ and, moreover, the nonincreasing of the zero number implies that $i \leq k$.

Notice however that, if one proves that the rescaled trajectory $\frac{u(t,.)}{\|u(t,.)\|}$ also converges in the C^1-norm to φ_i^t, then both the solution $u(t,.)$ and the eigenfunction φ_i^t must have the same zero number for all t sufficiently large, say $t \geq \tau$. Here $\tau > 0$ is such that the grow-up solution and the shifted grow-up solution share the same zero number. Therefore, once we prove that $\frac{u(t,.)}{\|u(t,.)\|}$ converges to φ_i^t in the C^1-norm, then we must have $i = k$. We focus now on proving the C^1 convergence, which will be done relying on the fact that $P_N D(A)$ has finite dimension and on the properties of Ψ.

The mapping Ψ is uniformly bounded in C^1, thus we get that, as $p(t,.)$ grows to infinity along with $u(t,.)$, $q(t,.)$ remains bounded. As a result, we have that

$$\lim_{t\to\infty} \frac{u(t,.)}{\|u(t,.)\|} = \lim_{t\to\infty} \frac{p(t,.)}{\|u(t,.)\|} = \lim_{t\to\infty} \frac{p(t,.)}{\|p(t,.)\|} = \lim_{t\to\infty} \frac{u(t,.)}{\|p(t,.)\|}.$$

Therefore,

$$\begin{aligned}
\lim_{t\to\infty} \|\frac{u(t,.)}{\|p(t,.)\|} - \varphi_i^t\|_{C^1} &= \lim_{t\to\infty} \|\frac{p(t,.)+q(t,.)}{\|p(t,.)\|} - \varphi_i^t\|_{C^1} \\
&\leq \lim_{t\to\infty} \|\frac{p(t,.)}{\|p(t,.)\|} - \varphi_i^t\|_{C^1} + \lim_{t\to\infty} \|\frac{q(t,.)}{\|p(t,.)\|}\|_{C^1} \\
&= \lim_{t\to\infty} \|\frac{p(t,.)}{\|p(t,.)\|} - \varphi_i^t\|_{C^1} + \lim_{t\to\infty} \frac{\|\Psi(p(t,.))\|_{C^1}}{\|p(t,.)\|} \\
&= \lim_{t\to\infty} \|\frac{p(t,.)}{\|p(t,.)\|} - \varphi_i^t\|_{C^1}.
\end{aligned}$$

Since $p(t,.)$ and $\varphi_i^t(.)$ are both in the finite-dimensional subspace $P_N D(A)$ and due to norm equivalence in finite dimension it follows that

$$\begin{aligned}
\lim_{t\to\infty} \|\frac{u(t,.)}{\|p(t,.)\|} - \varphi_i^t\|_{C^1} &\leq \lim_{t\to\infty} \|\frac{p(t,.)}{\|p(t,.)\|} - \varphi_i^t\|_{C^1} \\
&\leq \lim_{t\to\infty} C\|\frac{p(t,.)}{\|p(t,.)\|} - \varphi_i^t\| = 0,
\end{aligned}$$

since

$$\lim_{t\to\infty} \|\frac{u(t,.)}{\|u(t,.)\|} - \varphi_i^t\| = 0 \Longrightarrow \lim_{t\to\infty} \|\frac{p(t,.)}{\|p(t,.)\|} - \varphi_i^t\| = 0.$$

We then conclude that

$$\lim_{t\to\infty} \|\frac{u(t,.)}{\|u(t,.)\|} - \varphi_i^t\|_{C^1} = 0.$$

Finally, as we have mentioned before, the above equality implies that $i = k$ and thus

$$\lim_{t\to\infty} \|\frac{u(t,.)}{\|u(t,.)\|} - \varphi_k^t\|_{C^1} = 0.$$

□

4.2 Transfinite and intra-infinite heteroclinics

In order to approach the connections to infinity, we use the same method as that applied for a class of dissipative equations in [FR96] and for slowly non-dissipative equations in the form (4.2) with nonlinearities $g(x,u,u_x) \equiv g(u)$ in [BG11c]. The method consists of two steps: firstly one provides for any two distinct equilibria v and w sufficient conditions to prevent connections from v to w and then it is shown that connections which are not prevented in fact do exist.

At this point it is worth noticing that it does not exist a grow-up solution whose normalized trajectory converges to $\varphi_j^{\pm}$ if $j > \sqrt{b}$. This follows from Lemmas 2.3.1 and 2.3.2. As a consequence, the Poincaré projection of any grow-up solution can only converge to $\Phi_j^{\pm}$ if $j \in \{0, 1, ..., [\sqrt{b}]\}$.

Let E_f^c denote the set of equilibrium points of (2.1). Given $v \in E_f^c$, we say that v has a heteroclinic connection to the object at infinity $\Phi^\infty \in \{\Phi_j^{\infty,\pm} : j = 0, ..., [\sqrt{b}]\}$ if there exists a grow-up solution $u(t,.)$ satisfying

$$\lim_{t\to-\infty} u(t,.) = v \text{ and } \lim_{t\to\infty} \| \frac{u(t,.)}{\|u(t,.)\|} - \varphi(.) \|_{C^1} = 0,$$

where $\varphi \in \{\varphi_j^{\pm}\}_{j\in\mathbb{N}}$ and φ corresponds to the equilibria $\Phi \in \{\Phi_j^{\pm}\}_{j\in\mathbb{N}}$ on the sphere at infinity which is the projection of Φ^∞.

Taking into account the previous crucial results, we may now continue pursuing a complete decomposition of the non-compact global attractor $\mathcal{A}_f$. The next lemma determines when heteroclinic connections to infinity are blocked, i.e., when connections to an object at infinity are prevented. A necessary information to state the next result is that the eigenfunctions $\{\pm\varphi_j\}_{j\in\mathbb{N}}$ having only simple zeros implies that the zero number of the objects at infinity $\{\Phi_j^{\pm}\}_{j\in\mathbb{N}}$ are consistent, in the sense that the zero number of the corresponding heteroclinic orbits do not drop at $t = \infty$.

We define the zero number of $\Phi_j^{\infty,\pm}$ as

$$z(\Phi_j^{\infty,\pm}) = z(\varphi_j^{\pm}) = j \tag{4.3}$$

and

$$z(\Phi_j^{\infty,\pm} - v) = z(\Phi_j^{\infty,\pm}), \text{ for all } v \in E_f^c. \tag{4.4}$$

Also, we let

$$\mathrm{sign}(\Phi_j^{\infty,\pm}(0)) = \mathrm{sign}(\varphi_j^{\pm}(0)) = \pm 1 \tag{4.5}$$

and

$$\mathrm{sign}(\Phi_j^{\infty,\pm}(0) - v(0)) = \mathrm{sign}(\Phi_j^{\infty,\pm}(0)), \text{ for all } v \in E_f^c. \tag{4.6}$$

Under the above setting, we denote

$$\Phi_j^{\infty,-}(0) < v(0) < \Phi_j^{\infty,+}(0), \text{ for all } v \in E_f^c. \tag{4.7}$$

We then introduce the next definition, following the notion of adjacency introduced first in [Wol02].

Definition 4.2.1. *Let $v \in E_f^c$ and $\Phi^\infty \in \{\Phi_j^{\infty,\pm} : j = 0, ..., [\sqrt{b}]\}$ with $z(\Phi^\infty - v) = k$, for some $k \in \mathbb{N}$. We* *then say that v and Φ^∞ are k-adjacent if there is no $w \in E_f^c$ with $z(\Phi^\infty - w) = z(v - w) = k$ and*

(i) $v(0) < w(0) < \Phi^\infty(0)$*, if* $v(0) < \Phi^\infty(0)$

(ii) $\Phi^\infty(0) < w(0) < v(0)$*, if* $\Phi^\infty(0) < v(0)$*.*

Then, adapting the result of Infinite Blocking in [BG11c] to adjacency notion, we have the following.

Lemma 4.2.1 (Infinite Blocking Lemma). *Let $v \in E_f^c$ be a hyperbolic equilibrium and $\Phi^\infty \in \{\Phi_j^{\infty,\pm} : j = 0, ..., [\sqrt{b}]\}$ with $z(\Phi^\infty - v) = k$. If v and Φ^∞ are not k-adjacent, then v does not have any heteroclinic connection to the object Φ^∞ at infinity.*

Proof. Assume that there exists a grow-up solution $u(t,.)$ connecting v to Φ^∞. Since v and Φ^∞ are not k-adjacent, there exists $w \in E_f^c$ such that

$$z(\Phi^\infty - w) = z(v - w) = k \tag{4.8}$$

and

$$\Phi^\infty(0) < w(0) < v(0) \text{ or } v(0) < w(0) < \Phi^\infty(0). \tag{4.9}$$

If v connects to Φ^∞ via $u(t,.)$ then $\tilde{u} = u - w$ is a trajectory from $v - w$ to $\Phi^\infty - w = \Phi^\infty$ satisfying

$$\tilde{u}_t = \tilde{u}_{xx} + b\tilde{u} + \tilde{g}(x, \tilde{u}, \tilde{u}_x)$$

with $\tilde{g}(x, \tilde{u}, \tilde{u}_x) := g(x, w + \tilde{u}, w_x + \tilde{u}_x) - g(x, w, w_x)$. By Lemma 4.1.3 we may conclude that there exists $T > 0$ such that $z(u(T,.) - w(.)) = k$. Since (4.9) holds, the value of $w(0)$ lies between $u(T, 0)$ and $v(0)$, increasing the value of T if necessary.

Thus

$$u(T, 0) - w(0) < 0 < v(0) - w(0) \text{ or } v(0) - w(0) < 0 < u(T, 0) - w(0).$$

We have that $z(v - w) \geq z(u(T, .) - w)$ by the nonincreasing of $z(\tilde{u}(t, .))$. On the other hand, by (4.8)

$$z(v - w) = k = z(u(T, .) - w).$$

However, since $v(0) - w(0)$ and $u(T, .) - w(0)$ have opposite signs and $\tilde{u}(t, 0) \neq 0$ for all non-dropping times, we must have

$$z(v - w) > z(u(T, .) - w)$$

which is a contradiction. □

Notice that the previous result ensures that the existence of an equilibrium satisfying appropriate conditions may block heteroclinic connections to a class of equilibria at infinity. If there exists a bounded equilibrium w of equation (4.2) such that $z(v - w) = k$ then w blocks v from connecting to all objects $\Phi^\infty \in \{\Phi_j^{\infty,\pm} : j = 0, ..., [\sqrt{b}]\}$ at infinity as long as $z(\Phi^\infty) = k$ and

$$\Phi^\infty(0) < w(0) < v(0) \text{ or } v(0) < w(0) < \Phi^\infty(0).$$

The next lemma, on its turn, ensures the existence of connections whenever they are not blocked.

Lemma 4.2.2 (Infinite Liberalism Lemma)**.** *Let $v \in E_f^c$ be a hyperbolic equilibrium and $\Phi^\infty \in \{\Phi_j^{\infty,\pm} : j = 0, ..., [\sqrt{b}]\}$ with $z(\Phi^\infty - v) = k$. If v and Φ^∞ are k-adjacent, then v has a heteroclinic connection to the object Φ^∞ at infinity.*

Proof. We define $\iota := \mathrm{sign}(\Phi^\infty(0))$. Then choose

$$s_k := \iota \text{ and } \delta_j := \begin{cases} 0, \ j \geq k \\ \infty, \ j < k \end{cases}$$

and apply Lemma 3.0.5 to obtain an initial condition $u_0 \in W^u(v)$ such that the corresponding solution $u(t,.)$ satisfies

$$\begin{aligned} z(u(t,.) - v(.)) &= k \\ \iota &= \mathrm{sign}(u(t,0) - v(0)) \end{aligned}$$

for all $0 \leq t < \infty$. At first, we show that there does not exist any bounded equilibrium w of (4.2) such that $u(t,.)$ converges to w as t goes to infinity.

Let us assume that $\lim_{t\to\infty} u(t,.) = w$ for some bounded equilibrium w. Then, as the zero number of the shifted solution $u - v$ is nonincreasing

$$\lim_{t\to\infty} z(u(t,.) - v(.)) = z(w - v)$$

must be less or equal to k. Suppose $z(w - v) < k$. It would imply that the zero number of the shifted solution drops at infinity. We now show why this is not possible. We are assuming that $\lim_{t\to\infty} u(t,.) = w$, then

$$\lim_{t\to\infty} (u(t,.) - v(.)) = w - v =: \tilde{w}.$$

Since $\tilde{w} \not\equiv 0$, it follows that $\tilde{w}$ has only simple zeros, as it solves the ODE

$$0 = \tilde{u}_{xx} + b(\tilde{u}) + \tilde{g}(x, \tilde{u}, \tilde{u}_x).$$

Therefore, any solution in a sufficiently small neighborhood of $\tilde{w}$ must also have only simple zeros. It implies that $z(u(t,.) - v(.))$ is constant over some small neighborhood of $t = \infty$ and, thus,

$$z(u(t,.) - v(.)) = z(\tilde{w}(.)) = z(w - v) < k$$

in this neighborhood of $t = \infty$. However, if this is the case, then $z(u(t,.) - v(.))$ would have to drop at some finite time as $u(t,.)$ converges to w, which is a contradiction.

Therefore, if v connects to w via $u(t,.)$ then $z(w - v) = k$. Since $\iota = \mathrm{sign}(u(t,0) - v(0))$ for all $0 \leq t < \infty$, the sign of $u(t,0) - v(0)$ remains always positive or negative in forward time. Then, one should have

$$\operatorname{sign}(w(0)-v(0))=\iota.$$

However, there does not exist any bounded equilibrium w fulfilling these conditions, i.e,

$$z(w-v)=k \text{ and } \operatorname{sign}(w(0)-v(0))=\iota=\operatorname{sign}(\Phi^{\infty}(0)-w(0)),$$

since v and Φ^{∞} are k-adjacent.

Since $u(t,.)$ cannot converge to any bounded equilibrium, we conclude by Lemma 2.2.1 that

$$\lim_{t\to\infty}\|u(t,.)\|=\infty$$

and thus $u(t,.)$ is a grow-up solution. Finally, it was proven in Lemma 4.1.3 that in this case the normalized trajectory $\frac{u(t,.)}{\|u(t,.)\|}$ converges in the C^1-norm to the eigenfunction φ_k^{ι}. Which means that v has a heteroclinic connection to the object $\Phi_k^{\infty,\iota}$ at infinity. We then notice that $z(\Phi^{\infty})=z(\Phi^{\infty}-v)=k$ and $\operatorname{sign}(\Phi^{\infty}(0))=\iota$. It then follows from (4.3) and (4.5) that $\Phi^{\infty}=\Phi_k^{\infty,\iota}$. □

The former discussion provides us with a criterion to determine the existence of heteroclinics to the equilibria at infinity. We have then discussed the transfinite heteroclinic structure of $\mathcal{A}_f$, i.e., a description of the connections between the bounded subset $\mathcal{A}_f^c$ and a subset at infinity containing a finite number of equilibria at infinity. It remains to describe the connecting orbit structure within infinity. This has already been done in [Hel11], where there is a detailed discussion regarding the equilibria at infinity for equation (2.1).

To obtain the connections between the equilibria at infinity $\Phi_j^{\infty,\pm}$ we turn to equation (2.12). We first study the stability of the projections of these equilibria. If $j=0$, then

$$(\xi_n)_t=-n^2\xi_n, \text{ for } n\neq 0,$$

and therefore the equilibria $\Phi_0^{\pm}$ are stable. If $j\geq 1$ then

$$j^2-n^2>0, \text{ for } 0\leq n\leq j-1$$

$$j^2-n^2<0, \text{ for } n\geq j+1.$$

In this case, the equilibria $\Phi_j^{\pm}$ on the sphere at infinity have j unstable directions and infinitely many stable directions. Moreover, as proved in [Hel11], for each $j\in\mathbb{N}_0$, $\iota\in\{+1,-1\}$ and $n\neq j$ the ξ_n-axis is invariant and consists of heteroclinics from

$$\Phi_j^{\iota} \text{ to } \Phi_n^{\pm} \text{ if } n\leq j-1, \text{ and}$$

$$\Phi_n^{\pm} \text{ to } \Phi_j^{\iota} \text{ if } n\geq j+1.$$

We say that there exists an orbit connecting the equilibria at infinity $\Phi_i^{\infty,\iota}$ and $\Phi_j^{\infty,\pm}$ if the corresponding equilibria Φ_i^ι and $\Phi_j^\pm$ at the Poincaré equator $\mathcal{H}_e$, or sphere at infinity, are connected. Therefore the intra-infinite heteroclinics are namely the following:

$$\Phi_j^{\infty,\iota} \text{ connects to } \Phi_i^{\infty\pm}, \text{ for each } \iota \in \{-1,1\} \text{ and all } i \leq j-1. \tag{4.10}$$

4.3 Bounded equilibria

For the characterization of the non-compact global attractor $\mathcal{A}_f$ it remains to obtain the heteroclinic connections between the bounded equilibria. We then present an important discussion on such equilibria, obtaining results that are crucial for the next step which will be the description of the connections.

The set of bounded equilibria $E_f^c = \{v_1, ..., v_n\}$ of equation (2.1) is determined by the following second-order ODE

$$\begin{aligned} &u_{xx} + bu + g(x, u, u_x) = 0, \quad x \in (0, \pi) \\ &u_x = 0, \quad\ x = 0, \pi. \end{aligned} \tag{4.11}$$

We then associate with (4.11) the following initial value problem

$$\begin{aligned} &u_x = v \\ &v_x = -bu - g(x, u, v), \\ &u(0) = u_0, \quad v(0) = 0 \end{aligned} \tag{4.12}$$

We shall then remark that the set of solutions

$$u = u(x, u_0), v = v(x, u_0)$$

of (4.12) defines the two-dimensional manifold in $[0, \pi] \times \mathbb{R}^2$

$$L = \{(x, u, v)|\ u = u(x, u_0), v = v(x, u_0), u \text{ solves } (4.12), u_0 \in \mathbb{R},\ 0 \leq x \leq \pi\}.$$

Under the setting of equation (2.1), all trajectories of the initial value problem (4.12) are guaranteed to exist for all $x \in [0, \pi]$. Also, let the section curve of L at $x = \pi$ be denoted by γ, i.e.,

$$\gamma := \{(x, u(x, u_0), v(x, u_0)|\ u_0 \in \mathbb{R}, x = \pi\}. \tag{4.13}$$

One should then notice that there is a one-to-one correspondence between the equilibria of (2.1) and the intersection points of γ with the plane $v = 0$ (see [FR91, Roc91]).

We next take advantage of the just defined manifold L and obtain from its analysis that the equation (2.1) has at least one stationary solution. The assertion is already known for dissipative equations in the form (1.1). It follows from a series of results derived in [Zel68, Mat78, HM82]. In which concerns non-dissipative equations in the form (1.1), we know from [Mat79, Remark 4.6] that no satisfactory result was obtained except for the case $f(x, u, u_x) = f(u)$. Also, this state of affairs does not seem to have changed much in the last 30 years. Then, in the next lemma we address this matter and consider non-dissipative equations in the general form (2.1), under the nondegeneracy condition $b \neq n^2$.

Lemma 4.3.1. *If the dynamical system generated by equation (2.1) is*

(i) dissipative with $b < 0$, or

(ii) slowly non-dissipative with $b > 0$ and $b \neq n^2$ for $n \in \mathbb{N}$,

then the set of equilibria E_f^c is not empty.

Proof. To verify the non-emptiness of E_f^c we will prove that γ intersects the plane $v = 0$. We begin with the slowly non-dissipative case (ii). Rewriting (4.12) in the form

$$\begin{aligned} u_x &= \sqrt{b}w, \\ w_x &= -\sqrt{b}u - \frac{1}{\sqrt{b}}g(x, u, \sqrt{b}w) \\ u(0) &= u_0, \quad w(0) = 0. \end{aligned} \tag{4.14}$$

We then introduce in (4.14) the change to polar coordinates:

$$u = \rho\cos\theta, \quad w = -\rho\sin\theta, \tag{4.15}$$

with $\rho(0, u_0) = u_0$ and $\theta(0, u_0) = 0$. From (4.14) and (4.15), we obtain for $\rho \neq 0$ that

$$\begin{aligned} \rho_x &= \frac{2uu_x + 2ww_x}{2\rho} \\ &= \frac{1}{\sqrt{b}}g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\sin\theta. \end{aligned} \tag{4.16}$$

Moreover, from (4.14) and (4.15) we get

$$\begin{aligned} -\sqrt{b}\rho\sin\theta &= \rho_x\cos\theta - \rho\theta_x\sin\theta \\ -\sqrt{b}\rho\cos\theta - \frac{1}{\sqrt{b}}g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta) &= -\rho_x\sin\theta - \rho\theta_x\cos\theta \end{aligned} \tag{4.17}$$

Thus (4.17) implies

$$-\rho\theta_x(\sin^2\theta + \cos^2\theta) = -\sqrt{b}\rho(\sin^2\theta + \cos^2\theta) - \frac{1}{\sqrt{b}}g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\cos\theta.$$

One thus obtains

$$\theta_x = \sqrt{b} + \frac{1}{\sqrt{b}\rho}g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\cos\theta. \tag{4.18}$$

The function g is uniformly bounded with $|g(x,u,p)| < \Gamma$ for all x, u and p. We thus have that

$$\lim_{\rho\to\infty} \frac{1}{\sqrt{b}\rho} g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\cos\theta = 0,$$

i.e., for all $\delta > 0$ there exists $\hat{M} > 0$ large enough such that $|\rho(x)| > \hat{M}$ implies

$$\left|\frac{1}{\sqrt{b}\rho} g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\cos\theta\right| < \delta.$$

Moreover, we have

$$\begin{aligned}
\left|\int_0^x \rho_x(x)dx\right| &\leq \int_0^x |\rho_x(x)|\, dx \\
&= \int_0^x \left|\frac{1}{\sqrt{b}} g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\sin\theta\right| dx \\
&\leq \frac{1}{\sqrt{b}}\Gamma x \leq C,
\end{aligned}$$

for some constant C, which implies that

$$|\rho(x) - \rho(0)| = \left|\int_0^x \rho_x(x)dx\right| \leq C, \quad \forall x \in [0,\pi]. \tag{4.19}$$

From (4.19) we conclude that, for $|\rho(0)|$ sufficiently large, $|\rho(x)| > \hat{M}$ uniformly on $[0,\pi]$.

Therefore, if $|\rho(0)|$ is sufficiently large, we have

$$\left|\frac{1}{\sqrt{b}\rho} g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\cos\theta\right| < \delta$$

and thus, from (4.18) we have

$$\begin{aligned}
\left|\int_0^\pi (\theta_x - \sqrt{b})dx\right| &\leq \int_0^\pi \left|\theta_x - \sqrt{b}\right| dx \\
&= \int_0^\pi \left|\frac{1}{\sqrt{b}\rho} g(x, \rho\cos\theta, -\sqrt{b}\rho\sin\theta)\cos\theta\right| dx \\
&< \delta\pi.
\end{aligned}$$

Since $\theta(0) = 0$, we obtain

$$\left|\theta(\pi) - \sqrt{b}\pi\right| < \delta\pi.$$

Denoting $\varepsilon = \delta\pi$, we finally obtain that for all $\varepsilon > 0$ there exists $M > 0$ such that $|\rho(0)| > M$ implies

$$\left|\theta(\pi) - \sqrt{b}\pi\right| < \varepsilon. \tag{4.20}$$

As $|\rho(\pi)|$ grows with $|\rho(0)| = |u_0|$ and $b \neq n^2$ for $n \in \mathbb{N}$, we conclude from (4.20) that for both $\rho(0) > M$ and $\rho(0) < -M$, the curve γ is asymptotic to a straight line going from one semiplane of $\{(x, u, v) : x = \pi\}$ to the other. Therefore, the curve γ must intersect the plane $v = 0$, which implies that E_f^c is not empty.

For an idea of the proof of item (i) we refer the reader to [Zel68, Mat78, HM82]. □

The nondegeneracy condition $b \neq n^2$ in the previous lemma corresponds to the hyperbolicity of the equilibria at infinity, as we can see in [Hel11] and [BG11c]. This concept, as it is indicated in Section 2.3, relates to the projected equation obtained on the equator $\mathcal{H}_e$ from which we obtain the equilibria at infinity.

The previous lemma allows us to assume without loss of generality that

$$f(x, u, u_x) = bu + g(x, u, u_x) \in \mathcal{F}_0,$$

i.e., $f(x, 0, 0) = 0$. Indeed, the set of equilibria E_f^c being not empty allows us to introduce the change of variables $\tilde{u} = u - w$ for $w \in E_f^c$, obtaining (3.4) from equation (3.1).

Let $E_f^c = \{v_1, ..., v_n\}$ denote the set of equilibria of equation (2.1) ordered by their value at $x = 0$. We want now to determine the Morse index of v_1 and v_n by analysing the curve γ. We denote by $\overline{\theta} = \overline{\theta}(x, u_0)$ the solution of the differential equation

$$\overline{\theta}_x = \sqrt{b} + \frac{1}{\sqrt{b}} q(x, u_0) \cos^2 \overline{\theta} - r(x, u_0) \cos \overline{\theta} \sin \overline{\theta}, \;\; \overline{\theta}(0, u_0) = 0 \tag{4.21}$$

where $q(x, u_0) = g_u(x, u(x, u_0), v(x, u_0))$ and $r(x, u_0) = g_p(x, u(x, u_0), v(x, u_0))$. We also denote by $s(u_0) := (\pi, u(\pi, u_0), v(\pi, u_0))$ the point in the section curve γ corresponding to the initial condition u_0. Then $\overline{\theta}(\pi, u_0)$ can also be read as the angle swept clockwise by the unit vector tangent to γ at $s(u_0)$ as $s(u_0)$ describes γ, with u_0 going from $-\infty$ to ∞ (see [Roc85, Roc91]). Finally, let u_0^l be the initial value satisfying $u_0^l = v_l(0)$ for the equilibrium v_l. Then, the Morse index of v_l is given by

$$i(v_l) = 1 + [\overline{\theta}(\pi, u_0^l)/\pi],$$

[Roc85, Roc91].

Lemma 4.3.2. *Assume that all equilibria of (2.1) are hyperbolic. Then the Morse index of the minimal and maximal equilibria, v_1 and v_n, are given by*

$$i(v_1) = i(v_n) = 1 + [\sqrt{b}]. \tag{4.22}$$

Proof. The Morse index of an equilibrium v_l is given by

$$i(v_l) = 1 + [\overline{\theta}(\pi, u_0^l)/\pi], \;\; \forall \;\; l = 1, ..., n,$$

where $u_0^l = v_l(0)$. The equilibria v_1 and v_n being hyperbolic implies that

$$\overline{\theta}(\pi, u_0^1), \overline{\theta}(\pi, u_0^n) \neq k\pi, \;\; \forall \;\; k \in \mathbb{N}$$

(see [Roc85]). Let $\theta(\pi, u_0)$ be the angle swept clockwise by the position vector $(u(x, u_0), v(x, u_0))$ as x runs from $x = 0$ to $x = \pi$. It follows from the previous lemma that $\theta(x, u_0)$ satisfies

$$\theta_x = \sqrt{b} + \frac{1}{\rho\sqrt{b}} g(x, \rho\cos\theta, -\rho\sqrt{b}\sin\theta)\cos\theta$$

and

$$\lim_{|u_0|\to\infty} \theta(\pi, u_0) = \sqrt{b}\pi$$

(see (4.18) and (4.20)).

We affirm that

$$\lim_{|u_0|\to\infty} |\overline{\theta}(\pi, u_0) - \theta(\pi, u_0)| = 0. \tag{4.23}$$

To see this, we first modify the nonlinearity g as follows. We let $C_1 > 0$ be such that

$$|v_1(x)|, |v_n(x)| \leq C_1, \;\; \forall \;\; x \in [0, \pi].$$

Then, for u satisfying

$$|u(t, x)| \leq C_1$$

uniformly in (t, x), we have that

$$|u_x(t, x)| \leq C_2$$

uniformly in (t, x) for some constant C_2. This follows from the fact that $X^\alpha \subset C^1$ for $\alpha > 3/4$. We thus define

$$\tilde{g}(x, u, p) = \begin{cases} g(x, u, p), & \text{if } |u| \leq C_1, \; |p| \leq C_2 \\ 0, & \text{for } (u, p) \text{ outside } D, \end{cases} \tag{4.24}$$

where $D \subset \mathbb{R}^2$ is an open neighborhood of the set

$$\{(u, p) : |u| \leq C_1, |p| \leq C_2\}.$$

One should then verify that (4.24) does not alter the bounded subset $\mathcal{A}_f^c \subset \mathcal{A}_f$. For that, we notice the following. The dynamical system generated by (2.1) with g replaced by $\tilde{g}$ is non-dissipative and it possesses a non-compact global attractor

$$\tilde{\mathcal{A}}_f = \tilde{\mathcal{A}}_f^c \cup \tilde{\mathcal{A}}_f^\infty$$

where $\tilde{\mathcal{A}}_f^c$ is a bounded subset of $\tilde{\mathcal{A}}_f$. Moreover, $\mathcal{A}_f^c$ is contained in a sufficiently large ball $B \subset X^\alpha$.

Since $g = \tilde{g}$ on B, we obtain

$$\tilde{\mathcal{A}}_f^c \subset B,$$

by taking larger C_2 and C_1 if necessary. Therefore, $\tilde{\mathcal{A}}_f^c = \mathcal{A}_f^c$.

In what follows we consider (2.1) with g replaced by $\tilde{g}$. We want now to obtain the limit (4.23), i.e., we have to prove that for any $\epsilon > 0$ there exists $M > 0$ such that

$$|\overline{\theta}(x, u_0) - \theta(x, u_0)| < \epsilon,$$

if $|u_0| > M$, for $x = \pi$. For that we first let $x = 0$. We obtain $\overline{\theta}(0, u_0) = \theta(0, u_0) = 0$ and

$$\cos \overline{\theta}(0, u_0) = \cos \theta(0, u_0) = 1.$$

Consequently, there exists $s_1 > 0$ sufficiently small such that

$$|\cos \overline{\theta}(x, u_0) - \cos \theta(x, u_0)| < \epsilon_0, \;\; \forall \;\; 0 < x < s_1.$$

We thus have

$$|\cos \overline{\theta}(x, u_0)| \leq |\cos \overline{\theta}(x, u_0) - \cos \theta(x, u_0)| + |\cos \theta(x, u_0)| < \epsilon_0 + \overline{\epsilon}$$

if $0 < x < s_1$ and $|\cos \theta(x, u_0)| < \overline{\epsilon}$.

We notice that the solution u is given by

$$u(x, u_0) = \rho(x, u_0) \cos \theta(x, u_0).$$

Since $\lim\limits_{|u_0| \to \infty} \rho(x, u_0) = \infty$ then for $x \in [0, \pi]$ satisfying

$$|\cos \theta(x, u_0)| < \overline{\epsilon} \tag{4.25}$$

we have $(u(x, u_0), u_x(x, u_0)) \in D$, otherwise $(u(x, u_0), u_x(x, u_0)) \notin D$. Therefore, if (4.25) holds then

$$|\overline{\theta}_x(x, u_0)| \leq \sqrt{b} + \overline{\epsilon}$$

if $x \in (0, s_1)$, since $q(x, u_0)$ and $r(x, u_0)$ are bounded. If x does not satisfy (4.25) then

$$q(x, u_0) \equiv r(x, u_0) \equiv 0$$

and therefore $\overline{\theta}_x(x, u_0) \equiv \sqrt{b}$. We thus obtain the bound

$$
\begin{aligned}
|\overline{\theta}(s_1,u_0)-\theta(s_1,u_0)| &\le \int_0^{s_1} |\overline{\theta}_x(x,u_0)-\theta_x(x,u_0)|dx \\
&\le \int_0^{s_1} |\cos\overline{\theta}(x,u_0)||\frac{1}{\sqrt{b}}q\cos\overline{\theta}(x,u_0)-r\sin\overline{\theta}(x,u_0)|+|\theta_x(x,u_0)|dx \\
&\le (\epsilon_0+\overline{\epsilon}+\delta)\pi =: \epsilon_1,
\end{aligned}
$$

which implies that

$$
|\cos\overline{\theta}(s_1,u_0)-\cos\theta(s_1,u_0)| \le \epsilon_1.
$$

Then there exists $s_2 > s_1$ such that

$$
|\cos\overline{\theta}(s_1,u_0)-\cos\theta(s_1,u_0)| \le \epsilon_1, \ \ \forall \ \ s_1 < x < s_2
$$

and consequently

$$
\begin{aligned}
|\cos\overline{\theta}(x,u_0)| &\le |\cos\overline{\theta}(x,u_0)-\cos\theta(x,u_0)|+|\cos\theta(x,u_0)| \\
&\le \epsilon_1+\overline{\epsilon}
\end{aligned}
$$

if $x \in (s_1,s_2)$ and $|\cos\theta(x,u_0)| < \overline{\epsilon}$. By noticing that, if $x \in (s_1,s_2)$ is such that (4.25) does not hold then $\overline{\theta}_x(x,u_0) = \sqrt{b}$, we obtain as before the following bound

$$
\begin{aligned}
|\overline{\theta}(s_2,u_0)-\theta(s_2,u_0)| &\le \int_0^{s_2} |\overline{\theta}_x(x,u_0)-\theta_x(x,u_0)|dx \\
&\le (\epsilon_1+\overline{\epsilon}+\delta)\pi =: \epsilon_2.
\end{aligned}
$$

Then

$$
|\cos\overline{\theta}(s_2,u_0)-\cos\theta(s_2,u_0)| \le \epsilon_2
$$

and we obtain the existence of $s_3 > s_2$ such that

$$
|\cos\overline{\theta}(x,u_0)-\cos\theta(x,u_0)| \le \epsilon_2, \ \ \forall \ \ s_2 < x < s_3.
$$

We proceed with the argument as before until obtaining $s_n = \pi$, concluding that for $|u_0|$ sufficiently large

$$
|\overline{\theta}(\pi,u_0)-\theta(\pi,u_0)| \le \epsilon_n. \tag{4.26}
$$

Inequalities (4.26) and (4.20) imply that

$$
\lim_{|u_0|\to\infty} |\overline{\theta}(\pi,u_0)-\sqrt{b}\pi| = 0.
$$

We want finally to conclude that $[\overline{\theta}(\pi, u_0^n)/\pi] = [\sqrt{b}]$. Then, let γ_n be the arc described by

$$\{(\pi, u(\pi, u_0), u_x(\pi, u_0)) : u_0^n \leq u_0 \leq \rho\}$$

along $\gamma = \gamma(\pi)$, $\overline{\gamma}_n$ be the line segment

$$\{(\pi, u, 0) : u_0^n(\pi) < u < u(\pi, \rho)\}$$

and

$$\overline{\overline{\gamma}} = \{(\pi, u(\pi, \rho), p) : 0 \leq p < u_x(\pi, p)\}.$$

We then define

$$\Gamma_n = \gamma_n \cup \overline{\overline{\gamma}}_n \cup \overline{\gamma}_n.$$

Since v_n is the maximal equilibrium, $\gamma_n \cap \overline{\gamma}_n = \emptyset$. Moreover, $\gamma_n \cap \overline{\overline{\gamma}}_n = \emptyset$ by (4.20). We have then constructed a closed curve Γ_n which is piecewise differentiable and simple, i.e., a Jordan curve. It the follows from the turning tangent theorem applied to Γ_n that

$$\left[\frac{\overline{\theta}(\pi, u_0^n) - \overline{\theta}(\pi, \rho)}{\pi}\right] = 0.$$

Since $\overline{\theta}(\pi, u_0^n)/\pi \notin \mathbb{N}$ and $[\overline{\theta}(\pi, u_0^n)/\pi] = [\overline{\theta}(\pi, \rho)/\pi]$, then

$$[\overline{\theta}(\pi, \rho)/\pi] < \overline{\theta}(\pi, \rho)/\pi.$$

Therefore, since we can take ρ sufficiently large such that $\overline{\theta}(\pi, \rho)/\pi$ is sufficiently close to $\sqrt{b}$, we obtain

$$[\overline{\theta}(\pi, \rho)/\pi] = [\sqrt{b}].$$

Then, $[\overline{\theta}(\pi, u_0^n)/\pi] = [\sqrt{b}]$ and

$$i(v_n) = 1 + [\sqrt{b}].$$

$\square$

One of the crucial properties satisfied by the minimal and maximal equilibria of a dissipative system is that their Morse index are both equal zero. This property is fundamental if one wants to apply usual techniques to characterize the connections between the bounded equilibria. But, as we observed in (4.22), the Morse index of v_1 and v_n are both different from zero. In order to overcome this fact, we introduce a trick which will enable us to obtain the heteroclinic connections. This will be done in the next chapter.

Chapter 5

A permutation encoding the connections

For dissipative equations, some characterization results for the global attractors rely on the existence of a crucial associated permutation. See for instance [FR91, FR96, Wol02]. Many of the main geometric features of the global attractor are explicitly determined by this permutation. More precisely, for each equation in a very large class of dissipative equations, the permutation associated determines the Morse indices $i(v)$ and the intersection number $z(v-w)$ for all hyperbolic equilibria v, w, and these are precisely the necessary information to ascertain the connections among the equilibria.

A natural question arising at this point refers to the possibility of doing the same for slowly non-dissipative equations and obtain a simpler criterion to determine the heteroclinic connections. For a dissipative equation defined on a bounded interval $[x_1, x_2] \in \mathbb{R}$, the permutation associated to it consists of labeling the equilibria ordered firstly at $x = x_1$ and then at $x = x_2$. We thus affirm that one possible drawback of getting a related permutation for our equation (4.2) regards the objects playing the role of equilibria at infinity. Moreover, a permutation associated to a dissipative system satisfies specific properties which allow us to obtain the connections and, as one can expect, the permutation associated to our nonlinearity f does not satisfy such properties.

In order to overcome this scenario we introduce a crucial trick. It consists of a $([\sqrt{b}]+1)$-dimensionally unstable suspension, [FR00], of the bounded subset $\mathcal{A}_f^c$ of the non-compact global attractor $\mathcal{A}_f$. In the following we describe what we mean by this suspension and we show how we intend to do it.

The non-compact global attractor $\mathcal{A}_f$ contains bounded and unbounded equilibria. The set of equilibria of equation (2.1) is given by $E_f^c \cup E_f^\infty$. E_f^c is composed of n bounded equilibria and E_f^∞ by $2([\sqrt{b}]+1)$ equilibria at infinity. Roughly speaking, the suspension consists of joining to the equilibria $\{v_1, ..., v_n\}$ of (2.1) a set of $2([\sqrt{b}]+1)$ new bounded equilibria. We then try to associate each of the new equilibria with an equilibrium in E_f^∞. As a result, we will be able to work with only bounded equilibria and, moreover, the new equilibria will be added in a such a way that we obtain a dissipative system.

We begin with a discussion regarding the permutations we will associate to our problem (4.2). We then describe the suspension we intend to impose on $\mathcal{A}_f^c$ from which we obtain new equilibria corresponding to the equilibria at infinity. After that we write a characterization theorem prescribing the connections among all the bounded equilibria, the former and the new inserted ones, relying on the associated permutation. With the correspondence between the new bounded equilibria and the equilibria at infinity, we will finally be able to obtain a simple criterion for the existence of heteroclinic connections for our original problem (4.2).

5.1 Cut-off function

The starting point will be the bounded component of the non-compact global attractor. We aim at first to modify f outside a sufficiently large ball in order to obtain a dissipative semiflow. This must be done, however, in such a way that the modified semiflow coincides with the original one on the bounded subset $\mathcal{A}_f^c$ of $\mathcal{A}_f$. Therefore, a function h will be defined as follows. Recalling that $\mathcal{A}_f^c$ is contained in a large ball $B \subset X^\alpha$, we notice that on B

$$|u(t,x)| \leq C_1, \quad |u_x(t,x)| \leq C_2$$

uniformly in (t,x) for some constants C_1, C_2, due to the fact that $B \subset X^\alpha \subset C^1$. Let $D \subset \mathbb{R}^2$ be a sufficiently large ball such that it contains the set where $|u(t,x)| \leq C_1$ and $|u_x(t,x)| \leq C_2$. We let h be such that

$$h(x,u,p) = f(x,u,p), \text{ for all } (u,p) \in D$$

and for all $x \in [0,\pi]$. Later on we will define h for (u,p) outside D in such a way that the dynamical system induced by

$$\begin{cases} u_t = u_{xx} + h(x,u,u_x), \quad x \in [0,\pi] \\ u_x(t,0) = u_x(t,\pi) = 0. \end{cases} \tag{5.1}$$

is dissipative. Hence (5.1) will possess a compact global attractor $\mathcal{A}_h$. Nevertheless, we can decompose $\mathcal{A}_h$ into a maximal compact invariant subset $\mathcal{A}_h^c$ contained in B and its complement, that is,

$$\mathcal{A}_h = \mathcal{A}_h^c \cup \{\mathcal{A}_h \setminus \mathcal{A}_h^c\}, \text{ with } \mathcal{A}_h^c \subset B.$$

We notice that the set E_f^c of equilibria of (4.2) coincides with the subset of equilibria of equation (5.1) that are contained in B. This follows directly from the fact that $h = f$ on B. As a consequence, the set of all equilibria of (5.1) is decomposed as follows

$$E_h = E_f^c \cup \{E_h \setminus E_f^c\}.$$

The subset $\mathcal{A}_h^c \subset B$ is the set of all orbits that never leave B. Therefore, $\mathcal{A}_h^c$ is composed of the set E_f^c of equilibria in B and their heteroclinic orbits, which coincides with the definition of $\mathcal{A}_f^c$. In this way, the following lemma is verified.

Lemma 5.1.1. *Under the above notation, we have*

$$\mathcal{A}_f^c = \mathcal{A}_h^c.$$

Also let $\hat{D} \subset \mathbb{R}^2$ be a larger ball than D. We require that

$$h(x, u, p) = cu, \text{ for all } (u, p) \notin \hat{D},$$

for all $x \in [0, \pi]$ and for some constant $c < 0$. Further ahead we present the technical details for the definition of h in the remaining portion of the domain.

5.2 Permutations

As we have previously mentioned, there are permutations related to dissipative equations determining crucial information on the dynamics of the solutions. Namely, the heteroclinic connections in the global attractor for dissipative equations are determined by such permutations. If we suppose that the infinite dimensional dynamical system generated by equation (5.1) is dissipative, we are then able to recur to the well established literature addressing dissipative systems and their related permutations.

Let N be the number of equilibria for equation (5.1), i.e.,

$$N := \#E_h.$$

We also assume that all the equilibria are hyperbolic. The equilibria in E_h are denoted and labeled as

$$w_1(0) < w_2(0) < ... < w_N(0).$$

Let $\sigma_h \in S(N)$ be a permutation defined as follows

$$w_{\sigma_h(1)}(\pi) < w_{\sigma_h(2)}(\pi) < ... < w_{\sigma_h(N)}(\pi). \tag{5.2}$$

This permutation is related to the ordering of the points of intersection of the plane $v = 0$ with the curve γ_h defined for the stationary equation for (5.1) as in (4.13). The permutations related in this way to a Jordan curve like γ_h are called *meander permutations*.

We next recall some other definitions in the setting of dissipative equations. We say that a permutation $\sigma \in S(N)$ is a *dissipative permutation* if $\sigma(1) = 1$ and $\sigma(N) = N$. Given any $\sigma \in S(N)$ we also define the vector $(i_j(\sigma))_{1 \leq j \leq N}$ by

$$\begin{aligned} i_1(\sigma) &= 0 \\ i_{j+1}(\sigma) &= i_j(\sigma) + (-1)^{j+1} \operatorname{sign}(\sigma^{-1}(j+1) - \sigma^{-1}(j)) \end{aligned} \tag{5.3}$$

for $j = 1, ..., N-1$. Then if $i_j(\sigma) \geq 0$ for all $1 \leq j \leq N$, σ is called a *Morse permutation*. We say that the permutation σ is realizable by a dissipative equation (5.1) if $\sigma = \sigma_h$, with σ_h defined in (5.2). A permutation that is realizable in such way by a dissipative equation is referred to as a *Sturm permutation*.

For each equilibrium $w_j \in E_h$, the Morse index $i(w_j)$, i.e., the dimension of the corresponding unstable manifold $W^u(w_j)$, coincides with $i_j(\sigma_h)$. That is to say that

$$i(w_j) = i_j(\sigma_h), \text{ for all } j = 1, ..., N. \tag{5.4}$$

This fact is verified in [Roc85]. The indices $i_j(\sigma_h)$ can alternatively be written as

$$i_j(\sigma_h) = \sum_{m=1}^{j-1} (-1)^{m+1} \operatorname{sign}(\sigma_h^{-1}(m+1) - \sigma_h^{-1}(m)), \tag{5.5}$$

with empty sums denoting zero (see [FR96]). For dissipative systems, N is odd and the permutation defined as above is a dissipative Morse meander permutation. Moreover, a permutation related to dissipative systems satisfies (5.4) and we have $i(w_1) = i(w_N) = 0$.

When dealing with non-dissipative dynamical systems, we do not expect the above properties to hold. In fact, the dynamical system generated by equation (4.2) is non-dissipative and, as we verified in Lemma 4.3.2, the Morse indices of v_1 and v_n satisfy

$$i(v_1) = i(v_n) = [\sqrt{b}] + 1 =: k$$

with k strictly positive. This motivates the arguments and definitions introduced in the next section, where the non-dissipativity of (4.2) is highlighted.

5.3 Suspension

If dissipative properties are no longer verified, we are then led to expect that the Morse indices of the equilibria are not given by the expression in (5.5). In what follows we obtain the Morse indices of the equilibria $v_j \in E_f^c$ in terms of an associated permutation defined in an analogous manner as above.

We similarly define for equation (4.2) a permutation $\sigma_f \in S(n)$, where n denotes the number of bounded equilibria in E_f^c. The equilibria $v_1, v_2, ..., v_n \in E_c$ are labeled by their value at $x = 0$. We reorder them according to their values at $x = \pi$ and obtain $\sigma_f \in S(n)$ as

$$v_{\sigma_f(1)}(\pi) < v_{\sigma_f(2)}(\pi) < ... < v_{\sigma_f(n)}(\pi).$$

The permutation σ_f can also be defined from the corresponding section curve $\gamma = \gamma_f$ obtained for (2.1), as we can see in [Roc91]. Therefore σ_f is a meander permutation.

For this permutation $\sigma = \sigma_f \in S(n)$ we define the vector $(i_j(\sigma))_{1 \leq j \leq n}$ by

$$\begin{aligned} i_1(\sigma) &= k \\ i_{j+1}(\sigma) &= i_j(\sigma) + (-1)^{j+1+k}\operatorname{sign}(\sigma^{-1}(j+1) - \sigma^{-1}(j)), \end{aligned} \tag{5.6}$$

for $j = 1, ..., n-1$. We shall next verify that the dimension of the unstable manifold of the equilibrium v_j, which we are denoting by $i(v_j)$, coincides with $i_j(\sigma_f)$ for all $1 \le j \le n$.

Lemma 5.3.1. *Let $k = [\sqrt{b}] + 1$. If $\sigma = \sigma_f \in S(n)$ denotes the meander permutation corresponding to the section curve $\gamma = \gamma_f$ of (2.1) and $(i_j(\sigma_f))_{1\le j\le n}$ is the vector defined by (5.6), then*

$$i(v_j) = i_j(\sigma_f), \;\; 1 \le j \le n. \tag{5.7}$$

Proof. It was established in [Roc85] that the Morse indices $i(v_j)$ of the equilibria $v_j \in E_f^c$ are determined in terms of the curve γ defined in (4.13). As we mention in section 4.3, the relation is given by

$$i(v_j) = 1 + [\bar{\theta}(\pi, v_j(0))/\pi] \tag{5.8}$$

where $\bar{\theta}$ is the solution of (4.21). We now want to determine (5.8) explicitly in terms of σ_f.

Since $\bar{\theta}(\pi, v_j(0))$ can also be read as the angle swept clockwise by the unit vector tangent to γ at $s(u_0) = (\pi, u(\pi, u_0), p(\pi, u_0))$, with $p = u_x$, as u_0 goes from $-\infty$ to $v_j(0)$, we obtain from (5.8) that

$$i(v_{j+1}) = i(v_j) + \iota(j, j+1) \tag{5.9}$$

with $\iota(j, j+1) \in \{-1, +1\}$. From the alternative definition of $\bar{\theta}(\pi, v_j(0))$ in terms of γ_f and from (5.8), one obtains that

$$\iota(j, j+1) = \operatorname{sign}(v_{j+1}(\pi) - v_j(\pi))\operatorname{sign}(p_{u_0}(\pi, v_j(0))),$$

where p_{u_0} denotes the derivative of p with respect to u_0. It follows from the definition of σ_f that

$$\operatorname{sign}(v_{j+1}(\pi) - v_j(\pi)) = \operatorname{sign}(\sigma_f^{-1}(j+1) - \sigma_f^{-1}(j)). \tag{5.10}$$

We also have the relation

$$\operatorname{sign}(p_{u_0}(\pi, v_l(0))) = -\operatorname{sign}(p_{u_0}(\pi, v_{l+1}(0))), \text{ for all } 1 \le l \le n. \tag{5.11}$$

Moreover, the following inequalities hold

$$\begin{aligned} &p_{u_0}(\pi, v_1(0)) > 0, \text{ if } k \text{ is even, and} \\ &p_{u_0}(\pi, v_1(0)) < 0, \text{ if } k \text{ is odd.} \end{aligned}$$

These imply that $(-1)^k p_{u_0}(\pi, v_1(0)) > 0$ and, as a result, the alternation rule

$$(-1)^{j+1}(-1)^k p_{u_0}(\pi, v_j(0)) > 0$$

follows from (5.11). Which is equivalent to say that

$$\operatorname{sign}(p_{u_0}(\pi, v_j(0))) = (-1)^{j+1+k}. \tag{5.12}$$

We thus conclude from (5.10) and (5.12) that

$$\iota(j, j+1) = (-1)^{j+1+k} \operatorname{sign}(\sigma_f^{-1}(j+1) - \sigma_f^{-1}(j)),$$

i.e.,

$$i(v_{j+1}) = i(v_j) + (-1)^{j+1+k} \operatorname{sign}(\sigma_f^{-1}(j+1) - \sigma_f^{-1}(j)), \tag{5.13}$$

for all $1 \leq j \leq n$, as we wanted to prove. $\square$

Remark that by (4.22) and (5.7) we have

$$i_1(\sigma_f) = i_n(\sigma_f) = k.$$

In the sequel, we present the definition of suspension of $\sigma \in S(n)$.

Definition 5.3.1. *Let $\sigma \in S(n)$ denote a meander permutation and k a positive integer. We define the suspension $\hat{\sigma}^k$ of the permutation σ as the permutation $\hat{\sigma}^k \in S(n+2)$ which satisfies:*

(i) $\hat{\sigma}^k(j) = \sigma(j-1) + 1$, for $j \in \{2, ..., n+1\}$; and

(ii) if k is odd

$$\hat{\sigma}^k(1) = 1 \text{ and } \hat{\sigma}^k(n+2) = n+2\ ,$$

and if k is even

$$\hat{\sigma}^k(1) = n+2 \text{ and } \hat{\sigma}^k(n+2) = 1\ .$$

It is clear that $\hat{\sigma}^k$ is a meander permutation. For this permutation we define the vector $(i_j(\hat{\sigma}^k))_{1 \leq j \leq n+2}$ by (5.6) with k replaced by $k-1$. In particular, we have

$$i_1(\hat{\sigma}^k) = k - 1. \tag{5.14}$$

We thus conclude from (5.14) that after k suspensions of σ_f one obtains a meander permutation $\hat{\sigma}_f^1 \in S(n+2k)$ with a vector $(i_j(\hat{\sigma}_f^1))_{1 \leq j \leq n+2k}$ satisfying

$$i_1(\hat{\sigma}_f^1) = i_{n+2k}(\hat{\sigma}_f^1) = 0. \tag{5.15}$$

Indeed, since we are denoting by $\hat{\sigma}_f^j$ the suspension of $\hat{\sigma}_f^{j+1}$ and we have

$$i_1(\hat{\sigma}_f^j) = j - 1, \text{ for } 1 \leq j \leq k,$$

it follows that $i_1(\hat{\sigma}_f^1) = 0$. In order to prove that $i_{n+2k}(\hat{\sigma}_f^1) = 0$, the following should be verified

$$i_{n+2}(\hat{\sigma}^k) = k - 1.$$

We know that

$$i_{n+2}(\hat{\sigma}^k) = i_{n+1}(\hat{\sigma}^k) + (-1)^{n+2+k-1} \operatorname{sign}((\hat{\sigma}^k)^{-1}(n+2) - (\hat{\sigma}^k)^{-1}(n+1)),$$

by definition. It follows that

$$\begin{aligned} i_{n+2}(\hat{\sigma}^k) &= i_{n+1}(\hat{\sigma}^k) + \begin{cases} -\operatorname{sign}(n+2-(\hat{\sigma}^k)^{-1}(n+1)), & \text{if } k \text{ is odd} \\ \operatorname{sign}(1-(\hat{\sigma}^k)^{-1}(n+1)), & \text{if } k \text{ is even} \end{cases} \\ &= i_{n+1}(\hat{\sigma}^k) - 1, \end{aligned}$$

since $1 < (\hat{\sigma}^k)^{-1}(n+1) < n+2$. We shall next prove that $i_{n+1}(\hat{\sigma}^k) = k$. For that we need to verify the following

$$i_{j+1}(\hat{\sigma}^k) = i_j(\sigma), \text{ for all } 1 \leq j \leq n. \tag{5.16}$$

If $j = 1$, we have

$$\begin{aligned} i_2(\hat{\sigma}^k) &= i_1(\hat{\sigma}^k) + (-1)^{2+k-1} \operatorname{sign}((\hat{\sigma}^k)^{-1}(2) - (\hat{\sigma}^k)^{-1}(1)) \\ &= k - 1 + \begin{cases} \operatorname{sign}((\hat{\sigma}^k)^{-1}(2) - 1), & \text{if } k \text{ is odd} \\ -\operatorname{sign}((\hat{\sigma}^k)^{-1}(2) - (n+2)), & \text{if } k \text{ is even} \end{cases} \\ &= k - 1 + 1 = k = i_1(\sigma). \end{aligned}$$

Suppose that $i_j(\hat{\sigma}^k) = i_{j-1}(\sigma)$ for some $j \in \{1, ..., n-1\}$. Then

$$\begin{aligned} i_{j+1}(\hat{\sigma}^k) &= i_j(\hat{\sigma}^k) + (-1)^{j+1+k-1} \operatorname{sign}((\hat{\sigma}^k)^{-1}(j+1) - (\hat{\sigma}^k)^{-1}(j)) \\ &= i_{j-1}(\sigma) + (-1)^{j+k} \operatorname{sign}(1 + \sigma^{-1}(j) - (1 + \sigma^{-1}(j-1))) \\ &= i_{j-1}(\sigma) + (-1)^{j+k} \operatorname{sign}(\sigma^{-1}(j) - \sigma^{-1}(j-1)) \\ &= i_j(\sigma). \end{aligned}$$

We thus conclude that (5.16) holds and, consequently,

$$i_{n+1}(\hat{\sigma}^k) = i_n(\sigma) = k.$$

We then consider in more detail the permutation $\hat{\sigma}_f^1$. It is clear that $\hat{\sigma}_f^1 \in S(N)$ with $N = n + 2k$. Moreover, for all $j \in \{1, ..., k\} \cup \{n + k + 1, ..., n + 2k\}$, the permutation $\hat{\sigma}_f^1$ satisfies

$$\hat{\sigma}_f^1(j) = \begin{cases} j, \text{ for } j \text{ odd} \\ n + 2k - (j - 1), \text{ for } j \text{ even} \end{cases}, \tag{5.17}$$

Furthermore, by (i) of Definition 5.3.1, $\hat{\sigma}_f^1$ and σ_f are related by the following

$$\hat{\sigma}_f^1(k + l) = k + \sigma_f(l), \tag{5.18}$$

for all $l = 1, ..., n$. The next step is to verify the existence of a function h realizing the permutation $\hat{\sigma}_f^1$. Which is equivalent to say that there exists a function h such that $\hat{\sigma}_f^1$ is in fact the permutation obtained from the meander γ_h corresponding to h as defined in (5.2), i.e.,

$$\hat{\sigma}_f^1 = \sigma_h.$$

In order to do that we recall the following result, obtained in [FR99].

Proposition 5.3.1. *There exists a C^2 function h realizing the permutation $\sigma \in S(N)$ if, and only if, N is odd and σ is a dissipative Morse meander permutation.*

We already mentioned that the suspension process preserves the meander property of the permutation σ_f. Hence we have that $\hat{\sigma}_f^1$ is a meander permutation. Also, the permutation $\hat{\sigma}_f^1 \in S(N)$ is a dissipative permutation since $N = n + 2k$ is odd and, by (5.17),

$$\hat{\sigma}_f^1(1) = 1 \text{ and } \sigma_f^1(N) = N.$$

Moreover, the vector $(i_j(\sigma_f^1))_{1 \le j \le N}$ defined in (5.3) satisfies

$$i_j(\sigma_f^1) \ge 0, \tag{5.19}$$

for all $j = 1, ..., N$. Indeed, it follows from (5.15) that (5.19) holds for $j = 1$ and $j = N$. If $j \in \{k+1, ..., k+n\}$, we have from (5.16) that

$$i_{l+k}(\hat{\sigma}_f^1) = i_{l+k-1}(\hat{\sigma}_f^2) = ... = i_{l+1}(\hat{\sigma}_f^k) = i_l(\sigma) \ge 0,$$

for $l = 1, ..., n$. Lastly, if $j \in \{2, ..., k\} \cup \{n + k + 1, ..., n + 2k - 1\}$ we recur to the formula in (5.3). Hence,

$$\begin{aligned} i_{j+1}(\hat{\sigma}_f^1) &= i_j(\hat{\sigma}_f^1) + (-1)^{j+1} \operatorname{sign}((\hat{\sigma}_f^1)^{-1}(j+1) - (\hat{\sigma}_f^1)^{-1}(j)) \\ &= i_j(\hat{\sigma}_f^1) + \begin{cases} -\operatorname{sign}(j + 1 - (n + 2k - (j - 1))), & \text{for } j \text{ even} \\ \operatorname{sign}(n + 2k - j - j), & \text{for } j \text{ odd} \end{cases} \end{aligned}$$

if $j \in \{1, ..., k-1\} \cup \{n+k+1, ..., n+2k-1\}$, which is equivalent to

$$i_{j+1}(\hat{\sigma}_f^1) = i_j(\hat{\sigma}_f^1) + 1, \text{ for } j \in \{1, ..., k-1\}$$

and to

$$i_{j+1}(\hat{\sigma}_f^1) = i_j(\hat{\sigma}_f^1) - 1, \text{ for } j \in \{n+k+1, ..., n+2k-1\}.$$

Then, by recalling that $i_1(\hat{\sigma}_f^1) = i_{n+2k}(\hat{\sigma}_f^1) = 0$, it follows that

$$i_j(\hat{\sigma}_f^1) = j - 1 \geq 0$$
$$i_{n+k+j}(\hat{\sigma}_f^1) = k - j \geq 0,$$

for $j = 1, ..., k$. Therefore, $\hat{\sigma}_f^1$ is a Morse permutation. We are then able to apply the previous proposition and obtain the existence of a function h realizing $\hat{\sigma}_f^1$.

Next, we proceed by verifying that, in Proposition 5.3.1, it is possible to choose a function h satisfying the following requirements:

$$h(x, u, p) = f(x, u, p), \text{ for } (u, p) \in D \tag{5.20}$$

and

$$h(x, u, p) = cu, \text{ for } (u, p) \notin \hat{D}, \tag{5.21}$$

for all $x \in [0, \pi]$ and some $c < 0$. This is to say that we preserve the function f on the compact set $(x, u, p) \in [0, \pi] \times D$, and modifying it outside so that h becomes linearly dissipative outside a large set.

Before imposing on h the requirements (5.20) and (5.21) we remark that, on both the meanders $\gamma = \gamma_f$ and γ_h, the arcs joining the intersection points with the u-axis can be isotopically transformed into semicircles. Therefore, we are allowed to work with meanders γ in canonical form, i.e., essentially composed of semicircle arcs (see [FR99]).

The first requirement (5.20) is quite simple since we notice that γ_f and γ_h share the same permutation on the set E_f^c. We can then assume that $h(x, u, p) = f(x, u, p)$ for $(x, u, p) \in [0, \pi] \times D$.

The second requirement (5.21) follows from the realization results of canonical meanders by boundary value problems (see [FR99, FR91]), and the condition that for $|u|$ sufficiently large the projection $(u, p) \mapsto (u, 0)$ is a local diffeomorphism. We can thus assume that $h(x, u, p) = cu$ for $(u, p) \notin \hat{D}$, for some $c < 0$. We also have that the canonical meander γ_h, outside the semicircles shared with γ_f, is composed of semicircles corresponding to the suspension $\hat{\sigma}_f^k, \hat{\sigma}_f^{k-1}, ..., \hat{\sigma}_f^1$.

Therefore, the requirements (5.20) and (5.21) on h imply that the dynamical system induced by

$$\begin{cases} u_t = u_{xx} + h(x, u, u_x), & x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0. \end{cases}$$

is dissipative and preserves the equilibria in E_f^c and their heteroclinic connections. In particular, we conclude that the permutation $\sigma = \sigma_h$ is a Sturm permutation.

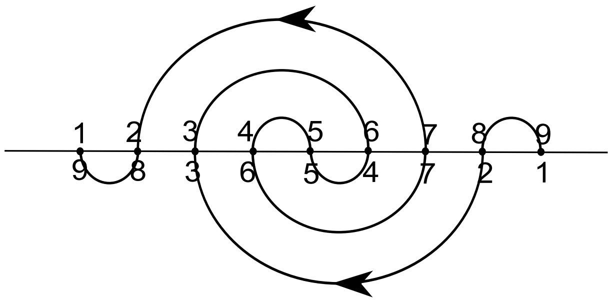

Figure 5.1: *Meander curve* γ_f *for* $f(x,u,u_x) = 10u + 16\sin u.$

In order to illustrate the suspension method we consider an example. If we set the function f to be

$$f(x,u,u_x) = 10u + 16\sin u,$$

we obtain from the initial value problem (4.12) the canonical form of the curve γ_f, as depicted in Fig. 5.1. Moreover, a numerical computation of a time map related to (4.12), performed in [BG11c, Section 6], leads us to

$$\sigma_f = (9\ \ 8\ \ 3\ \ 6\ \ 5\ \ 4\ \ 7\ \ 2\ \ 1).$$

Hence, the 4-th suspension of σ_f is given by

$$\hat{\sigma}_f^1 = (1\ \ 16\ \ 3\ \ 14\ \ 13\ \ 12\ \ 7\ \ 10\ \ 9\ \ 8\ \ 11\ \ 6\ \ 5\ \ 4\ \ 15\ \ 2\ \ 17),$$

as in Fig. 5.2. The solid line in Fig. 5.3 is associated with the permutation σ_f, while the dashed line represents the suspension $\hat{\sigma}_f^1$ of σ_f. We notice that, in this particular case, $n = 9$ and $k = 4$. Furthermore, the computed Morse vector related to $\hat{\sigma}_f^1$ is given by

$$(i_j(\hat{\sigma}_f^1))_{1\leq j\leq 17} = (0,1,2,3,4,3,4,5,6,5,4,3,4,3,2,1,0),$$

as indicated in Fig. 5.3.

The permutation $\sigma_h = \hat{\sigma}_f^1$ being Sturm implies that the Morse indices $i(w_j)$ can be obtained in terms of σ_h as in (5.5). Moreover, in this case, the intersection numbers $z(w_j - w_m)$ are also given explicitly in terms of σ_h, for all $j, m \in \{1, ...N\}$, as we can see in [FR91, Roc91, FR96] and as it is presented in the next proposition.

Proposition 5.3.2. *Under the above setting and considering the Sturm permutation* $\sigma_h \in S(N)$*, for* $1 \leq m < l \leq N$*, the Morse indices are given by*

$$i(w_m) = \sum_{j=1}^{m-1} (-1)^{j+1} \operatorname{sign}(\sigma_h^{-1}(j+1) - \sigma_h^{-1}(j))$$

and the intersection numbers by

$$z(w_l - w_m) = i(w_m) + \frac{1}{2}[(-1)^l \operatorname{sign}(\sigma_h^{-1}(l) - \sigma_h^{-1}(m)) - 1]$$

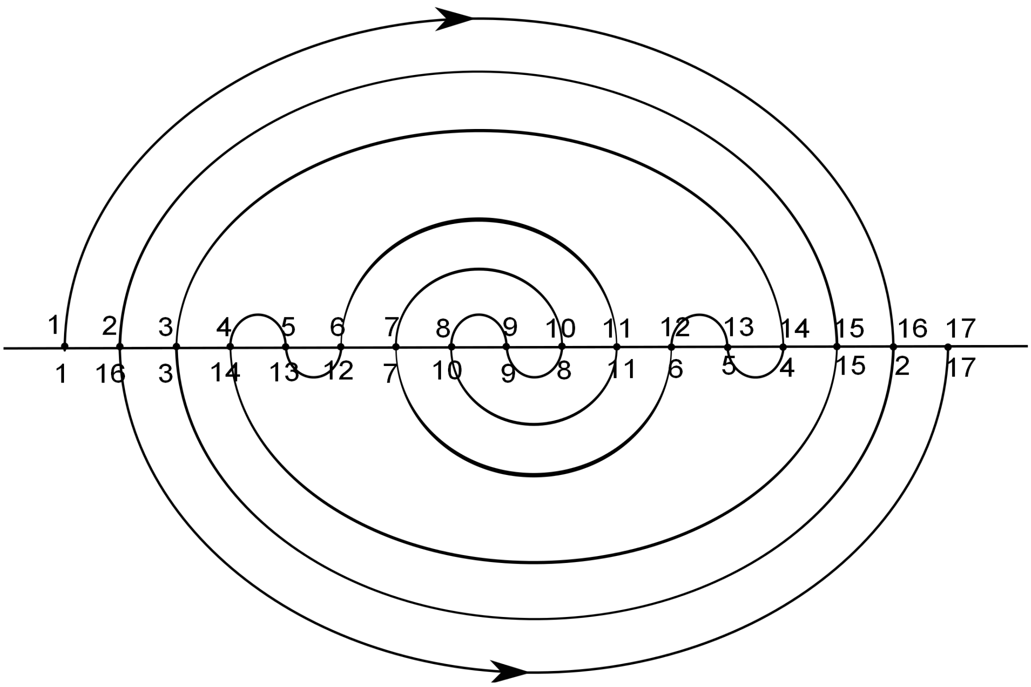

Figure 5.2: *Meander curve with permutation* $\hat{\sigma}_f^1$ *for* $f(x, u, u_x) = 10u + 16 \sin u$.

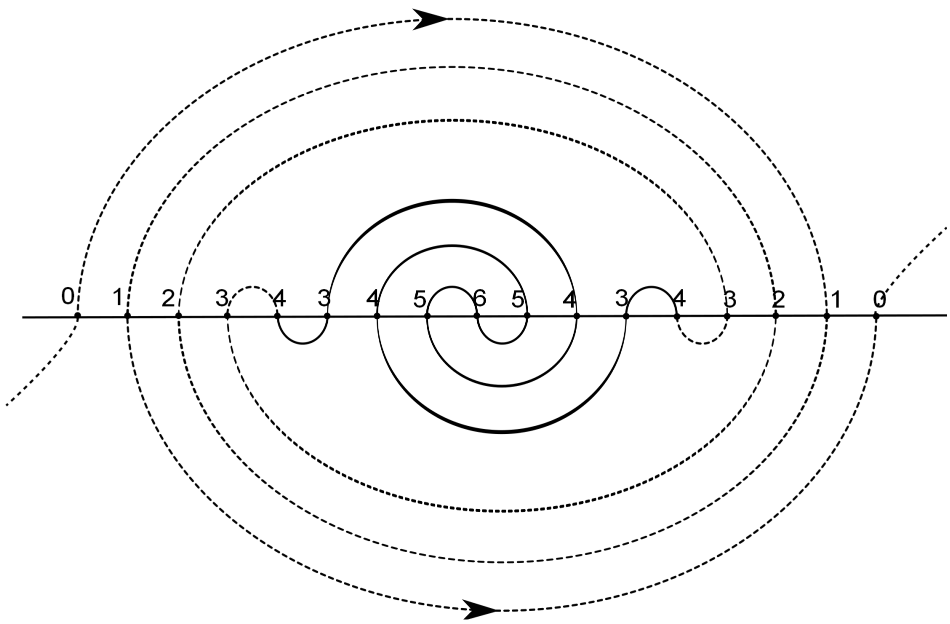

Figure 5.3: *Suspension of the meander curve* γ_f *for* $f(x, u, u_x) = 10u + 16 \sin u$.

$$+\sum_{j=m+1}^{l-1}(-1)^j \operatorname{sign}(\sigma_h^{-1}(j)-\sigma_h^{-1}(m)),$$

where empty sums denote zero.

We are then allowed to apply Proposition 5.3.2 to obtain explicit expressions for the Morse indices $i(w_j)$ and the zero numbers $z(w_j - w_m)$, for all $j, m \in \{1, ..., n+2k\}$, in terms of $\sigma_h = \hat{\sigma}_f^1$, or equivalently, use the recursion formulas in [FR96, Porposition 2.1]. We collect in the following lemmas the results previously obtained from $\hat{\sigma}_f^1$ for the Morse indices $i(w_j)$, $w_j \in E_h$.

Lemma 5.3.2. *The Morse indices of the exterior equilibria in E_h, i.e., the equilibria $w_j \in E_h$ with $j \in \{1, ..., k\} \cup \{n+k+1, ..., n+2k\}$, satisfy:*

$$i(w_j) = j - 1, \; for\ 1 \leq j \leq k,$$

and

$$i(w_{n+2k-j}) = j, \; for\ 0 \leq j \leq k-1.$$

Lemma 5.3.3. *The Morse indices of the interior equilibria in E_h, i.e., the equilibria $w_j \in E_h$ with $j \in \{k+1, ..., k+n\}$, satisfy:*

$$i(w_{k+l}) = i(v_l), \; for\ 1 \leq l \leq n.$$

Hence, the Morse indices $i(v_l)$ and $i(w_{k+l})$ coincide for all $1 \leq l \leq n$. This follows from (5.16) and is also a direct consequence of (5.20). The above lemma displays a relation between the Morse indices $i(w_j)$, $w_j \in E_h$, and the Morse indices $i(v_l)$, $v_l \in E_f^c$.

Regarding the information on the zero numbers contained on the permutation σ_h we have two results. In Lemma 5.3.4 we present the zero numbers results on the intersection of the k first and last equilibria with the remaining n middle equilibria in E_h, i.e, the intersection of the exterior with the interior equilibria. In Lemma 5.3.5 we obtain the zero number results for the n middle equilibria $w_j \in E_h$, i.e, the interior equilibria. We also notice that these results coincide with those for the equilibria $v_l \in E_f^c$.

Lemma 5.3.4. *For any $1 \leq l \leq n$, the following holds*

$$z(w_{k+l} - w_j) = j - 1, \; for\ 1 \leq j \leq k \tag{5.22}$$

$$z(w_{n+2k-j} - w_{k+l}) = j, \; for\ 0 \leq j \leq k-1. \tag{5.23}$$

Proof. By [FR96, Proposition 2.1] we are also provided with the following expressions

$$z(w_{j+1} - w_j) = \min\{i(w_{j+1}), i(w_j)\}, \quad 1 \le j \le n + 2k \tag{5.24}$$

$$z(w_{l+1} - w_j) = z(w_l - w_j) + \frac{1}{2}[(-1)^{l+1} \operatorname{sign}(\sigma_h^{-1}(l+1) - \sigma_h^{-1}(j)) \tag{5.25}$$
$$+ (-1)^l \operatorname{sign}(\sigma_h^{-1}(l) - \sigma_h^{-1}(j))], \quad 1 \le j < l \le n + 2k - 1$$

$$z(w_j - w_1) = z(w_{n+2k} - w_j) = 0, \quad 2 \le j \le n + 2k - 1. \tag{5.26}$$

In order to prove the first statement of the lemma we claim that, for any $1 \le j \le k$

$$z(w_{k+l} - w_j) = z(w_{k+1} - w_j), \quad \text{for } 2 \le l \le n. \tag{5.27}$$

Indeed, from (5.25) one gets the equality

$$z(w_{k+l} - w_j) = z(w_{k+l-1} - w_j) + \frac{1}{2}[(-1)^{k+l} \operatorname{sign}(\sigma_h^{-1}(k+l) - \sigma_h^{-1}(j))$$
$$+ (-1)^{k+l-1} \operatorname{sign}(\sigma_h^{-1}(k+l-1) - \sigma_h^{-1}(j))]$$

and the computation should then be split into two cases:

$$\operatorname{sign}(\sigma_h^{-1}(k+l) - \sigma_h^{-1}(j)) = \operatorname{sign}(\sigma_h^{-1}(k+l-1) - \sigma_h^{-1}(j)) = -1,$$

for j even, and

$$\operatorname{sign}(\sigma_h^{-1}(k+l) - \sigma_h^{-1}(j)) = \operatorname{sign}(\sigma_h^{-1}(k+l-1) - \sigma_h^{-1}(j)) = +1,$$

for j odd. This follows from the fact that

$$\sigma_h^{-1}(k+l) \in \{k+1, ..., k+n\} \text{ for all } 1 \le l \le n. \tag{5.28}$$

But, since $(-1)^{k+l} = -(-1)^{k+l-1}$, in both cases one should obtain

$$\frac{1}{2}[(-1)^{k+l} \operatorname{sign}(\sigma_h^{-1}(k+l) - \sigma_h^{-1}(j))$$
$$+ (-1)^{k+l-1} \operatorname{sign}(\sigma_h^{-1}(k+l-1) - \sigma_h^{-1}(j))] = 0.$$

Therefore, $z(w_{k+l} - w_j) = z(w_{k+l-1} - w_j)$ for all $1 \le j \le k$ and $2 \le l \le n$. In particular, we obtain the claim (5.27).

It is then sufficient to calculate $z(w_{k+1} - w_j)$. For that we write

$$z(w_{j+m} - w_j) = z(w_{j+m-1} - w_j) + \frac{1}{2}[(-1)^{j+m} \operatorname{sign}(\sigma_h^{-1}(j+m) - \sigma_h^{-1}(j))$$
$$+ (-1)^{j+m-1} \operatorname{sign}(\sigma_h^{-1}(j+m-1) - \sigma_h^{-1}(j))],$$

for $2 \leq m \leq k-j$ and $1 \leq j \leq k$. We further notice that

$$\operatorname{sign}(\sigma_h^{-1}(j+m) - \sigma_h^{-1}(j)) = \operatorname{sign}(\sigma_h^{-1}(j+m-1) - \sigma_h^{-1}(j)) = -1,$$

for j even, and

$$\operatorname{sign}(\sigma_h^{-1}(j+m) - \sigma_h^{-1}(j)) = \operatorname{sign}(\sigma_h^{-1}(j+m-1) - \sigma_h^{-1}(j)) = +1,$$

for j odd, by (5.17). As before, we obtain

$$z(w_{j+m} - w_j) = z(w_{j+m-1} - w_j) \tag{5.29}$$

for $2 \leq m \leq k-j$, since $(-1)^{j+m} = -(-1)^{j+m-1}$. For $m = k+1-j$ we have

$$\begin{aligned} z(w_{k+1} - w_j) = z(w_k - w_j) + \frac{1}{2}[(-1)^{k+1} \operatorname{sign}(\sigma_h^{-1}(k+1) - \sigma_h^{-1}(j)) \\ + (-1)^k \operatorname{sign}(\sigma_h^{-1}(k) - \sigma_h^{-1}(j))] \end{aligned}$$

which, along with (5.28), imply that

$$z(w_{k+1} - w_j) = z(w_k - w_j).$$

Therefore we have that (5.29) holds for $2 \leq m \leq k+1-j$. We thus conclude that

$$z(w_{k+1} - w_j) = z(w_{j+1} - w_j).$$

By (5.25) and the previous lemma we conclude that $z(w_{k+1} - w_j) = j-1$. Hence, (5.27) implies (5.22) for $1 \leq l \leq n$.

To guarantee (5.23), we first prove that

$$i(w_{k+l}) = k - \sum_{m=l+1}^{n} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) \tag{5.30}$$

for all $1 \leq l \leq n$. We should remark that for $l = n$ the sum above is empty, which means that $i(w_{k+n}) = k$, as we expect it to be. In order to prove (5.30), we first notice the following:

$$z(w_{n+2k} - w_{k+l}) = 0,$$

by (5.26), and

$$\begin{aligned} z(w_{n+2k} - w_{k+l}) = i(w_{k+l}) + \frac{1}{2}[(-1)^{n+2k} \operatorname{sign}(\sigma_h^{-1}(n+2k) - \sigma_h^{-1}(k+l)) - 1] \\ + \sum_{m=l+1}^{n+k-1} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)), \end{aligned}$$

by Proposition 5.3.2. The above equalities imply

$$i(w_{k+l}) = 1 - \sum_{m=l+1}^{n+k-1} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)), \tag{5.31}$$

since $(-1)^{n+2k} \operatorname{sign}(\sigma_h^{-1}(n+2k) - \sigma_h^{-1}(k+l)) - 1 = -2$. Therefore,

$$\begin{aligned} i(w_{k+l}) =& 1 - \sum_{m=l+1}^{n} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) \\ & - (-1)^{k+n+1} \operatorname{sign}(\sigma_h^{-1}(k+n+1) - \sigma_h^{-1}(k+l)) - ... - \\ & - (-1)^{n+2k-1} \operatorname{sign}(\sigma_h^{-1}(n+2k-1) - \sigma_h^{-1}(k+l)) \\ =& 1 - \sum_{m=l+1}^{n} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) - (-1)(k-1) \\ =& k - \sum_{m=l+1}^{n} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)). \end{aligned}$$

This shows (5.30).

We are then able to obtain (5.23) using (5.30). It follows from Proposition 5.3.2 that

$$\begin{aligned} z(w_{n+2k-j} - w_{k+l}) =& i(w_{k+l}) + \frac{1}{2}[(-1)^{n+2k-j} \operatorname{sign}(\sigma_h^{-1}(n+2k-j) - \sigma_h^{-1}(k+l)) - 1] \\ & + \sum_{m=l+1}^{n+k-j-1} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) \\ =& i(w_{k+l}) - 1 + \sum_{m=l+1}^{n} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) + \\ & + (-1)^{k+n+1} \operatorname{sign}(\sigma_h^{-1}(k+n+1) - \sigma_h^{-1}(k+l)) + ... + \\ & + (-1)^{n+2k-j-1} \operatorname{sign}(\sigma_h^{-1}(n+2k-j-1) - \sigma_h^{-1}(k+l)) \\ =& i(w_{k+l}) - 1 + \sum_{m=l+1}^{n} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) + \\ & + (-1)(k-j-1) \\ =& i(w_{k+l}) + \sum_{m=l+1}^{n} (-1)^{k+m} \operatorname{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) - k + j \\ =& k - k + j = j, \end{aligned}$$

for $0 \leq j \leq k-1$ and $1 \leq l \leq n$. Then (5.23) holds for $1 \leq l \leq n$. ☐

We finally present the next result on the zero number of the interior equilibria in E_h. The next lemma follows from (5.20). The result also provides a relation between the zero numbers of the equilibria $w_j \in E_h$, with $j \in \{k+1, ..., k+n\}$, and the equilibria $v_l \in E_f^c$.

Lemma 5.3.5. *For all $1 \leq r < l \leq n$, the following holds:*

$$z(w_{k+l} - w_{k+r}) = z(v_l - v_r). \tag{5.32}$$

The next result, along with the above lemma, provides us with an expression for $z(w_{k+l} - w_{k+r})$ in terms of σ_f.

Lemma 5.3.6. *For any $1 \leq r < l \leq n$, the zero number $z(v_l - v_r)$ is given by*

$$\begin{aligned} z(v_l - v_r) = i(v_r) + \frac{1}{2}[(-1)^{l+k} \operatorname{sign}(\sigma_f^{-1}(l) - \sigma_f^{-1}(r)) - 1] \\ + \sum_{j=r+1}^{l-1} (-1)^{j+k} \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)). \end{aligned}$$

Proof. We let δ_l denote the unit tangent vector to γ_f at $s(v_l(0))$ and c_{rl} denote the winding of γ_f around $(v_l(\pi), 0) \in \gamma_f$, that is, the number of half-turns swept clockwise by the curve γ_f around $(v_l(\pi), 0)$, from $v_r(0)$ to $v_l(0)$. We then know from [Roc91] that

$$z(v_l - v_r) = i(v_r) + c_{rl} \tag{5.33}$$

if δ_l belongs to the odd quadrants, and

$$z(v_l - v_r) = i(v_r) + c_{rl} - 1 \tag{5.34}$$

if δ_l belongs to the even quadrants. Hence, it follows from an argument similar to that used in Lemma 5.3.1 that

$$z(v_l - v_r) = i(v_r) + \sum_{j=r+1}^{l-1} (-1)^{j+k} \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r))$$

if δ_l belongs to the odd quadrants, and

$$z(v_l - v_r) = i(v_r) + \sum_{j=r+1}^{l-1} (-1)^{j+k} \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)) - 1$$

if δ_l belongs to the even quadrants.

Moreover, if δ_l belongs to the odd quadrants then

$$\operatorname{sign}(p_{u_0}(\pi, v_l(0))) = -1 \text{ and } \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)) = -1$$

or

$$\operatorname{sign}(p_{u_0}(\pi, v_l(0))) = +1 \text{ and } \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)) = +1.$$

Which is equivalent to say that

$$\operatorname{sign}(p_{u_0}(\pi, v_l(0))) \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)) = +1.$$

If δ_l belongs to the odd quadrants then

$$\operatorname{sign}(p_{u_0}(\pi, v_l(0))) = -1 \text{ and } \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)) = +1$$

or

$$\operatorname{sign}(p_{u_0}(\pi, v_l(0))) = +1 \text{ and } \operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)) = -1.$$

Which is equivalent to say that

$$\operatorname{sign}(p_{u_0}(\pi, v_l(0)))\operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)) = -1.$$

We also have, as in Lemma 5.3.1, that $\operatorname{sign}(p_{u_0}(\pi, v_l(0))) = (-1)^{k+l}$. We thus obtain

$$\begin{aligned} z(v_l - v_r) = i(v_r) &+ \frac{1}{2}[(-1)^{l+k}\operatorname{sign}(\sigma_f^{-1}(l) - \sigma_f^{-1}(r)) - 1] \\ &+ \sum_{j=r+1}^{l-1}(-1)^{j+k}\operatorname{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(r)), \end{aligned}$$

as we wanted. □

Remarkably, Lemmas 5.3.2 and 5.3.4 yield the following convenient relations, for $0 \leq j \leq k-1$:

$$\begin{aligned} i(w_{j+1}) &= i(\Phi_j^{\infty,-}) \\ i(w_{n+2k-j}) &= i(\Phi_j^{\infty,+}) \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} z(w_{k+l} - w_{j+1}) &= z(v - \Phi_j^{\infty,-}) \\ z(w_{n+2k-j} - w_{k+l}) &= z(\Phi_j^{\infty,+} - v) \end{aligned} \tag{5.36}$$

for all $1 \leq l \leq n$, for any bounded equilibrium $v \in E_f^c$ and any equilibria at infinity $\Phi_j^{\infty,\pm}$. Moreover, as seen in Lemmas 5.3.3 and 5.3.5, the Morse indices and zero numbers for the interior equilibria $w_j \in E_h$, with $j \in \{k+1, ..., k+n\}$, coincide with the corresponding Morse indices and zero numbers for the equilibria $v_l \in E_f^c$.

In the next section we use the above relations to obtain all the heteroclinic connections on the non-compact global attractor.

5.4 Heteroclinic connections

Under the setting of equation (5.1), we recall the following definition and proposition from the theory of dissipative systems, as we can see in [Wol02].

Definition 5.4.1. *Consider any two equilibria w_r and w_m in E_h with $z(w_r - w_m) = j$ and $w_m(0) < w_r(0)$.*

We say that w_r and w_m are j-adjacent if there does not exist any other equilibrium $w \in E_h$ satisfying

$$z(w_r - w) = z(w - w_m) = j$$

and

$$w_m(0) < w(0) < w_r(0).$$

Proposition 5.4.1 ([Wol02]). *Let w_r and w_m be equilibria in E_h with $z(w_r - w_m) = j$. Then there exists a heteroclinic connection between w_r and w_m if, and only if, they are j-adjacent.*

It is verified in Lemma 5.1.1 that the bounded subset $\mathcal{A}_f^c$ of $\mathcal{A}_f$ coincides with the subset $\mathcal{A}_h^c \subset \mathcal{A}_h$, where $\mathcal{A}_h^c \subset B$ and $\mathcal{A}_h = \mathcal{A}_h^c \cup \{\mathcal{A}_h \setminus \mathcal{A}_h^c\}$. We know, in particular, that the equilibrium $w_{k+l} \in E_h$ coincides with the equilibrium $v_l \in E_f$, for any $l \in \{1, ..., n\}$. Moreover, the heteroclinic connections in $\mathcal{A}_h^c$ between the equilibria w_{k+l} are identical to the heteroclinic connections in $\mathcal{A}_f^c$ between the equilibria v_l. All of the above facts follow from (5.20). We thus conclude that the connections on $\mathcal{A}_f^c$ are given as in Proposition 5.4.1, i.e., there is a heteroclinic connection between v_l and v_m if, and only if, v_l and v_m are adjacent. Then σ_h determines the connecting orbit structure on $\mathcal{A}_f^c$.

Regarding the connections with the equilibria at infinity we have the following. Suppose that an equilibrium $w_{k+l} = v_l$, for some $l \in \{1, ..., n\}$, has a heteroclinic connection to the equilibrium w_j, for some $j \in \{1, ..., k\}$. We thus know, from Lemma 5.3.4, that

$$z(w_{k+l} - w_j) = j - 1.$$

Since w_{k+l} connects to w_j, we conclude from Proposition 5.4.1, that there does not exist any equilibrium $w \in E_h$ satisfying

$$z(w_{k+l} - w) = z(w - w_j) = j - 1 \tag{5.37}$$

and

$$w_j(0) < w(0) < w_{k+l}(0). \tag{5.38}$$

We want to see that the equilibrium $v_l = w_{k+l}$ connects to the equilibrium in E_f that corresponds to w_j, that is, the equilibrium at infinity $\Phi_{j-1}^{\infty,-}$. Assume by way of contradiction that this is not the case. Then, it follows from Lemma 4.2.2 that there exists an equilibrium $v \in E_f^c$ satisfying

$$z(v_l - v) = z(v - \Phi_{j-1}^{\infty,-}) \tag{5.39}$$

and

$$\Phi_{j-1}^{\infty,-}(0) < v(0) < v_l(0). \tag{5.40}$$

Which leads to a contradiction because $v = w_{k+m}$ for some $m \in \{1, ..., n\}$ and, then, v could not satisfy (5.39) and (5.40) since there does not exist $w \in E_h$ satisfying (5.37) and (5.38).

We thus conclude that the heteroclinic connections to infinity in $\mathcal{A}_f$ are also given as in Proposition 5.4.1, that is to say that the permutation σ_h determines the heteroclinic connections to infinity in $\mathcal{A}_f$.

Theorem 5.4.1. *Let u and v be equilibria in E_f satisfying $z(u-v)=j$, with the extended interpretation of the zero number for the equilibria at infinity given by (4.3) and (4.4). Then there exists a heteroclinic connection between u and v if, and only if, they are j-adjacent. Moreover, if the equilibria v and w are connected, then the one with higher Morse index is the source of the connection.*

Chapter 6

Conclusion

We have considered the slowly non-dissipative system generated by equation (4.2). We have obtained that the associated non-compact global attractor $\mathcal{A}_f$ is composed of a compact subset $\mathcal{A}_f^c$ and an unbounded subset $\mathcal{A}_f^\infty$. We have noticed that $\mathcal{A}_f^c$ is contained in a sufficiently large ball $B \in X^\alpha$ and it comprises the bounded equilibria and their heteroclinic connections. The unbounded subset $\mathcal{A}_f^\infty$, on its turn, contains the equilibria at infinity and the grow-up solutions. From the existence of an inertial manifold containing the non-compact global attractor, we have obtained Lemmas 4.2.1 and 4.2.2 which determine the heteroclinic connections between the bounded equilibria in E_f^c and the equilibria at infinity E_f^∞, i.e., the transfinite heteroclinics. Moreover, (4.10) describes the heteroclinic connections within infinity, as obtained in [Hel11].

By considering a suspension of the permutation σ_f related to our equation (4.2), we have obtained the associated dissipative equation (5.1). We have constructed the nonlinearity h of the obtained dissipative equation in such a way that it coincides with f in the large ball $B \in X^\alpha$. Since the set of bounded equilibria E_f^c is contained in B, it coincides with the set of equilibria of equation (5.1) that are contained in B. Therefore, since the permutation σ_h determines, through the adjacency notion, the heteroclinic connections between the equilibria in E_h of equation (5.1), the permutation σ_h also determines the heteroclinic connections between the equilibria in E_f^c of equation (4.2). By gathering the above mentioned results, we have obtained all the heteroclinic connections on the non-compact global attractor $\mathcal{A}_f$.

We have further made a correspondence between the equilibria of our slowly non-dissipative equation (4.2) and the equilibria of the associated dissipative equation (5.1). We have then proved that this correspondence preserves the Morse indices and the zero numbers of the difference between the equilibria. Given that, we verified that the correspondence also preserves the connections between the equilibria. Since the permutation σ_h determines the Morse indices and the zero numbers of the equilibria of equation (4.2) we have then concluded that σ_h determines, also through the notion of adjacency, the heteroclinic connections on the non-compact global attractor.

It is worth noticing that our main result also holds for the case $f = f(u)$. We have then generalized the main Theorem in [BG10] to include the case of the more general nonlinearity $f = f(x, u, u_x)$. We have also provided a much simpler criterion for describing the connections than that appearing in [BG10].

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