# Definable endofunctions in uniformly locally finite structures

 Vinicius Cifú Lopes

We prove that definable endofunctions in uniformly locally finite structures enjoy a form of the pigeonhole principle.

**Surjectivity-injectivity of definable endofunctions.** Fix a structure $\mathcal{A}$ in some first-order, many-sorted language, and write $\mathrm{Def}(\mathcal{A})$ for the category of sets and functions which are definable with parameters in $\mathcal{A}$. Two properties of endomorphisms (that is, arrows of the form $X \to X$) in $\mathrm{Def}(\mathcal{A})$ can be of interest:

(S⇒I) Every surjective definable endofunction is injective.

(I⇒S) Every injective definable endofunction is surjective.

We first observe that (S⇒I) implies (I⇒S), and they are equivalent in the presence of definable Skolem functions. For, let $g\colon X \to X$ be definable and injective; take $f\colon X \to X$, $f(x) = g^{-1}(x)$ if $x \in g[X]$, or $f(x) = x$ otherwise; then $f$ is definable and surjective, so it is injective by (S⇒I); use that to show $g$ surjective. Next, let $f\colon X \to X$ be definable and surjective; take a definable "choice map" $g\colon X \to X$, i.e., $g(x) \in f^{-1}[x]$ for every $x \in X$; then $g$ is injective, so it is surjective by (I⇒S); hence $f$ is injective.

One can show that (I⇒S) is equivalent to a weak form of additive cancellation in the Grothendieck semiring of $\mathrm{Def}(\mathcal{A})$. This is the collection of classes of isomorphism of the category, endowed with natural operations of addition and multiplication.

Transfering the relevant question to a finite structure, as we will do below, is the same approach that one can take to investigate for pseudofiniteness. In fact, (S⇒I) holds in pseudofinite structures, and J. Ax, *Injective endomorphisms of varieties and schemes,* Pacific J. Math. 31 (1969), pp. 1–7, has famously used pseudofiniteness to show that algebraically closed fields satisfy (I⇒S).

We establish (S⇒I) for the eq-expansion of any vector space over any division ring, and more generally (I⇒S) for any strongly minimal group, in *Euler characteristics for strongly minimal groups and the eq-expansions of vector spaces,* J. Symbolic Logic 76 (2011), pp. 235–242.

**In uniformly locally finite structures.** Recall that a structure $\mathcal{B}$ is *uniformly locally finite* if any finitely generated substructure of $\mathcal{B}$ is finite and its cardinality is bounded by a natural number depending only on $\mathcal{B}$ and the cardinality of the generator set.

We will prove: "Assume that any structure which is definable in $\mathcal{A}$ is uniformly locally finite. Then the property (S⇒I) holds in $\mathcal{A}$."

Start with $f\colon X \to X$ which is surjective and definable in $\mathcal{A}$. Then the structure $(X, f)$, with one sort and one unary function symbol, is uniformly locally finite. In particular, there is $N \in \mathbb{N}$ such that any $x \in X$ satisfies $|\langle x\rangle| \leqslant N$. Recall that, in this case, $\langle x\rangle = \{\, f^k(x) \mid k \in \mathbb{N}\,\}$.

Given $p \in X$, we will first prove that $f^{-1}[p] \subseteq \langle p\rangle$. Fix $x_0 \in f^{-1}[p]$ and $x_{n+1} \in f^{-1}[x_n]$ for $n \in \mathbb{N}$. Since $\langle x_N\rangle$ has at most $N$ elements, but it contains $x_0, \ldots, x_N$, there are $0 \leqslant n < m \leqslant N$ with $x_n = x_m$. Take $k = m - n - 1 \geqslant 0$, thus $x_k = f^{n+1}(x_m) = f^{n+1}(x_n) = p$, and hence $x_0 = f^k(x_k) = f^k(p) \in \langle p\rangle$.

Now we have $\langle p\rangle \subseteq f[\langle p\rangle]$ by iterations of $f$, thus $f[\langle p\rangle] = \langle p\rangle$. We obtained a surjective endofunction $f|_{\langle p\rangle}\colon \langle p\rangle \to \langle p\rangle$ on a finite set, hence it must be injective.

Finally, suppose there are $a, b \in X$ with $f(a) = f(b)$. Take $p = f(a)$. Then $a, b \in f^{-1}[p] \subseteq \langle p\rangle$, and $f|_{\langle p\rangle}$ is injective, so $a = b$. This finishes the proof.

**Conclusion.** In turn, if we consider a structure in which all interpretable structures are uniformly locally finite, then every surjective interpretable endofunction is injective. For example, every structure interpretable in an $\omega$-categorical structure is again $\omega$-categorical, and $\omega$-categorical structures are uniformly locally finite, see Cor. 7.3.2 in W. Hodges, *Model Theory,* Cambridge Univ. Press, 1993.

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